1. Fubini and Tonelli.

(a) Let $X$ be a positive random variable (i.e, $X \geq 0$ a.s). Show that

$$E(X) = \int_0^\infty P(X > t) \, dt$$

(note $X$ need not have an absolutely-continuous distribution). Also verify that for any $\alpha > 0$

$$E(X^\alpha) = \alpha \int_0^\infty t^{\alpha-1}P(X > t) \, dt$$

(b) Define probability spaces $(\Omega_i, \mathcal{B}_i, \mu_i)$, for $i = 1, 2$ as follows. Let each $\Omega_i := (0,1)$, the unit interval, with $\sigma$-algebras

$$\mathcal{B}_1 = \text{Borel sets of } (0,1) \quad \mathcal{B}_2 = \text{All subsets of } (0,1),$$

and let $\mu_1$ be Lebesgue measure and $\mu_2$ counting measure– so that $\mu_1(A)$ is the length of any set $A \in \mathcal{B}_1$ and $\mu_2(A)$ is the number of points in $A \in \mathcal{B}_2$. Define

$$f(x, y) = 1_{x=y}(x, y)$$

Set

$$I_1 = \int_{\Omega_1} \left[ \int_{\Omega_2} f(x, y) \mu_2(dy) \right] \mu_1(dx) \quad I_2 = \int_{\Omega_2} \left[ \int_{\Omega_1} f(x, y) \mu_1(dx) \right] \mu_2(dy)$$

Compute $I_1$ and $I_2$. Is $I_1 = I_2$? Are the measures $\mu_1$ and $\mu_2$ $\sigma$-finite? Why doesn’t Fubini’s theorem hold here?

(c) This problem is a probabilistic version of the familiar integration-by-parts formula from calculus. Suppose $F$ and $G$ are two distribution functions with no common points of discontinuity in an interval $(a, b)$. Show that

$$\int_{(a,b]} G(x)F(dx) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} F(x)G(dx)$$

Show that the above formula need not hold true if $F$ and $G$ have common discontinuities.

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2. Uniform Integrability (UI).

(a) Let \( \{X_n\} \) be a sequence of iid, \( L_1 \) random variables. Set \( S_n \equiv \sum_{i=1}^{n} X_i \). Show that the sequence of random variables \( \{Y_n\} \) defined by \( Y_n \equiv S_n/n \) is UI.

(b) Let \( X_n \sim \text{No}(0, \sigma_n^2) \). Find a simple (easily verifiable) condition on \( \{\sigma_n^2\} \) such that \( \{X_n\} \) is UI.

(c) If \( \{X_n\} \) and \( \{Y_n\} \) are UI, show that so is \( \{X_n + Y_n\} \).

(d) Suppose \( \{X_n, n \geq 1\} \) is an arbitrary sequence of non-negative random variables, and set \( M_n \equiv \vee_{i=1}^{n} X_i \). If \( \{X_n\} \) is UI, show that \( \mathbb{E}(M_n)/n \to 0 \).

(e) Let \( \phi(x) \) be a function which grows faster than \( x \) at infinity, i.e, \( \phi(x)/x \to \infty \) as \( x \to \infty \). Let \( \mathcal{C} \) be a collection of random variables such that, for some fixed \( B < \infty \) and all \( Z \in \mathcal{C} \),
\[
\mathbb{E}(\phi(|Z|)) \leq B.
\]
Show that \( \mathcal{C} \) is UI.

3. Convergence Theorems Revisited.

(a) Let \( X \) be a non-negative real valued random variable. Show that:
\[\begin{align*}
\text{i. } \lim_{n \to \infty} n \mathbb{E} \left( \frac{1}{n} \mathbf{1}_{[X > n]} \right) &= 0. \\
\text{ii. } \lim_{n \to \infty} n^{-1} \mathbb{E} \left( \frac{1}{n} \mathbf{1}_{[X > n^{-1}]} \right) &= 0.
\end{align*}\]

(b) Suppose \( \{p_k, k \geq 0\} \) is a probability mass function on \( \{0, 1, \ldots\} \) and define the generating function
\[P(z) = \sum_{k=0}^{\infty} p_k z^k \quad 0 \leq z \leq 1\]
Prove using Dominated Convergence theorem that
\[\frac{d}{dz} P(z) = \sum_{k=1}^{\infty} p_k k z^{k-1} \quad 0 \leq z \leq 1.\]
Note you may wish to consider the cases \( z < 1 \) and \( z = 1 \) separately. What is \( P'(1) \)? \( P'(0) \)? Can you express the variance of \( X \), if it exists, in terms of \( P(z) \) and its derivatives?