Final Examination

STA 215: Statistical Inference

Tuesday, 2006 May 2, 2:00 – 5:00 pm

This is a closed-book examination; please put all your books and notes on the floor. You may use a calculator, but no other electronic device.

A normal distribution table, a distribution handout, and a blank worksheet are attached to the exam. Ask me if you want more scratch paper.

If a question seems ambiguous or confusing please ask me instead of guessing.

You must show your work to get partial credit. Unsupported answers are not acceptable, even if they are correct.

<table>
<thead>
<tr>
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<th>/20</th>
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<tbody>
<tr>
<td>1.</td>
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<td>Total:</td>
<td>/100</td>
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Print Name: ______________________
Problem 1. The value $B_t$ of Brownian motion at time $t > 0$ has a Gaussian distribution $B_t \sim \mathcal{N}(0, t)$; the length of time $X \equiv \min\{t : B_t \geq a\}$ until it attains level $a > 0$ has a distribution called the “Inverse Gaussian,” denoted $\text{IG}(a)$, with density function

$$f(x \mid a) = ae^{-a^2/2x} / \sqrt{2\pi x^3}, \quad x > 0.$$ 

Let $\bar{x} = (x_1, \cdots, x_n)$ be the observed value of a random sample $\{X_j\} \overset{iid}{\sim} \text{IG}(a)$.

a) Show that $\text{IG}(a)$ is an Exponential Family. Find the dimension $q$, the natural parameter $\eta(a) \in \mathbb{R}^q$, sufficient statistic $T(x)$, and normalizing constant $B(a)$ in the standard form

$$f(x \mid a) = \exp\{\eta(a) \cdot T(x) - B(a)\} h(x).$$

b) Find the Maximum Likelihood Estimator $\hat{a}(\bar{x})$ for a random sample of size $n$.

$$\hat{a}(\bar{x}) = \cdots$$

c) Does $X \sim \text{IG}(a)$ have a finite mean? Finite variance? Why?

(XC) For which $p \in \mathbb{R}$ is $E[X^p] < \infty$?
Problem 2. The observations \( \{X_j\} \overset{iid}{\sim} \text{Pa}(\theta) \) for \( 1 \leq j \leq n \) follow a Pareto distribution with p.d.f.

\[
f(x | \theta) = \theta x^{-\theta}, \quad x \geq 1
\]

for some uncertain \( \theta \in \Theta = [0, \infty) \) (note the range \( x \in X = [1, \infty) \)).

a) Find the MLE \( \hat{\theta} \) for the parameter \( \theta \in \Theta = \mathbb{R}_+ \):

\[
\hat{\theta}(\bar{x}) = 
\]

b) Find the Fisher Information for a single observation:

\[
I(\theta) = 
\]

c) Is this an Exponential Family? If so, find the dimension \( q \), the natural parameter \( \eta(\theta) \in \mathbb{R}^q \), sufficient statistic \( T(x) \), and normalizing constant \( B(\theta) \) in the standard form

\[
f(x | \theta) = \exp \left\{ \eta(\theta) \cdot T(x) - B(\theta) \right\} h(x);
\]

if not, explain why.
**Problem 3.** With the same Pareto Pa(\(\theta\)) model as above, \(X \sim \theta x^{-1-\theta}\) for \(x \geq 1\), and a random sample \(\bar{x} \in \mathcal{X} = [1, \infty)^n\) of size \(n\),

a) Find the posterior distribution \(\pi_{\alpha,\beta}(\theta | \bar{x})\) (by name or p.d.f.) and the posterior mean \(\bar{\theta}_{\alpha,\beta}(\bar{x})\) for a Bayesian analysis with a gamma prior distribution \(\theta \sim \text{Ga}(\alpha, \beta)\), i.e., prior p.d.f. \(\pi_{\alpha,\beta}(\theta) = \theta^{\alpha-1}/\Gamma(\alpha)\), \(\theta > 0\):

\[
\pi_{\alpha,\beta}(\theta | \bar{x}) = \quad \quad \bar{\theta}_{\alpha,\beta}(\bar{x}) = \quad \quad
\]

b) Find the posterior distribution \(\pi_J(\theta | \bar{x})\) and the posterior mean \(\bar{\theta}_J(\bar{x})\) for a reference Bayesian analysis using the Jeffreys prior distribution:

\[
\pi_J(\theta | \bar{x}) = \quad \quad \bar{\theta}_J(\bar{x}) = \quad \quad
\]
Problem 3 (cont).

c) Find the conditional CDF and, for Gamma prior distribution $\theta \sim \text{Ga}(\alpha, \beta)$, the marginal CDF for an observation $X \sim \text{Pa}(\theta)$:

\[
F(x \mid \theta) = P[X \leq x \mid \theta] = \text{ } \\
F_{\alpha,\beta}(x) = P_{\alpha,\beta}[X \leq x] = \text{ }
\]

d) With the Jeffreys prior distribution, if we make the single observation of $X_1 = 1.10$, what is the conditional probability that the next observation, $X_2$, will exceed 10? (Hint: This is easy from Parts b) and c) above)

\[
P_J[X_2 > 10 \mid X_1 = 1.10] = \text{ }
\]
Problem 4. With the same Pareto Pa(θ) model as above, \( X \sim \theta x^{-1-\theta} \) for \( x \geq 1 \), and with a single observation \( X = x \in \mathcal{X} = [1, \infty) \) (a random sample of size \( n = 1 \)),

a) Find the rejection region \( R \) for the most powerful possible test of the hypotheses

\[
H_0 : \theta = 1 \quad \text{vs.} \quad H_1 : \theta = 20
\]

with size \( \alpha = 0.10 \), and evaluate its power. Show your work.

\[ R = \quad \text{1 - } \beta = \quad \]

b) Find an exact symmetric 80% confidence interval \([L_x, R_x]\) for \( \theta \). Evaluate \( L_x \) and \( R_x \) numerically for \( x = 1.10 \).
Problem 4 (cont).

STILL with the same Pareto Pa(θ) model \( X \sim \theta x^{-1-\theta} \) for \( x \geq 1 \), and the single observation \( X = x \in \mathcal{X} = [1, \infty) \),

c) Find the \( P \)-value for testing \( H_0 : \theta = 1 \) vs. \( H_1 : \theta = 20 \) if we observe \( X = 1.10 \). Can you reject \( H_0 \) at level \( \alpha = 0.10 \)?

\[ P = \text{} \quad \text{Reject?} \quad \bigcirc \text{Yes} \quad \bigcirc \text{No} \quad (\text{at } \alpha = 0.10). \]

d) You COULD have answered the “Reject or not?” question of Part c) above from EITHER your answer to Part a) OR your answer to Part b) above. How?
Problem 5. The fun never stops. We still have data \( \{X_i\} \) iid \( \text{Pa}(\theta) \), with p.d.f. \( X \sim \theta x^{-1-\theta} \) for \( x \geq 1 \), with \( \theta \in \Theta \equiv \mathbb{R}_+ \).

a) Find the expected value \( E[X_i \mid \theta] \) for all \( \theta \in \Theta \) (be careful!):
\[
E[X_i \mid \theta] = \quad \text{__________________________}
\]

b) Find the Method of Moments estimator for \( \theta \), based on a sample \( \bar{x} \in \mathcal{X} = [1, \infty)^n \) of size \( n \).
\[
\hat{\theta}_{\text{MOM}}(\bar{x}) = \quad \text{__________________________}
\]

c) Explain why \( \hat{\theta}_{\text{MOM}} \) cannot possibly achieve the minimum possible squared-error risk among estimators of \( \theta \).
Extra worksheet, if needed (ask if you’d like more):
<table>
<thead>
<tr>
<th>Name</th>
<th>Notation</th>
<th>pdf/pmf</th>
<th>Range</th>
<th>Mean $\mu$</th>
<th>Variance $\sigma^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beta</td>
<td>$\text{Be}(\alpha, \beta)$</td>
<td>$f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$</td>
<td>$x \in (0, 1)$</td>
<td>$\frac{\alpha}{\alpha+\beta}$</td>
<td>$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$</td>
</tr>
<tr>
<td>Binomial</td>
<td>$\text{Bi}(n, p)$</td>
<td>$f(x) = \binom{n}{x} p^x q^{n-x}$</td>
<td>$x \in 0, \cdots, n$</td>
<td>$np$</td>
<td>$npq$ \quad $(q = 1-p)$</td>
</tr>
<tr>
<td>Exponential</td>
<td>$\text{Ex}(\lambda)$</td>
<td>$f(x) = \lambda e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$1/\lambda$</td>
<td>$1/\lambda^2$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\text{Ga}(\alpha, \lambda)$</td>
<td>$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\alpha/\lambda$</td>
<td>$\alpha/\lambda^2$</td>
</tr>
<tr>
<td>Geometric</td>
<td>$\text{Ge}(p)$</td>
<td>$f(x) = pq^x \quad f(y) = pq^{y-1}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$q/p$</td>
<td>$q/p^2$ \quad $(q = 1-p)$</td>
</tr>
<tr>
<td>HyperGeo.</td>
<td>$\text{HG}(n, A, B)$</td>
<td>$f(x) = \frac{\binom{A}{x} \binom{B}{n-x}}{\binom{A+B}{n}}$</td>
<td>$x \in 0, \cdots, n$</td>
<td>$nP$</td>
<td>$nP (1-P) \frac{N-n}{N-1}$ \quad $(P = \frac{A}{A+B})$</td>
</tr>
<tr>
<td>Logistic</td>
<td>$\text{Lo}(\mu, \beta)$</td>
<td>$f(x) = e^{-(x-\mu)/\beta} / [1 + e^{-(x-\mu)/\beta}]^2$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\pi^2 \beta^2 / 3$</td>
</tr>
<tr>
<td>Log Normal</td>
<td>$\text{LN}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi} \sigma^2} e^{-(\log x - \mu)^2 / 2\sigma^2}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$e^{\mu+\sigma^2/2}$</td>
<td>$e^{2\mu+\sigma^2} (1 - e^{-\sigma^2})$</td>
</tr>
<tr>
<td>Neg. Binom.</td>
<td>$\text{NB}(\alpha, p)$</td>
<td>$f(x) = \binom{\alpha-1}{x} p^\alpha q^{x} \quad f(y) = \binom{\alpha-1}{y} p^\alpha q^{y-\alpha}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\alpha q/p$</td>
<td>$\alpha q/p^2$ \quad $(q = 1-p)$</td>
</tr>
<tr>
<td>Normal</td>
<td>$\text{No}(\mu, \sigma^2)$</td>
<td>$f(x) = \frac{1}{\sqrt{2\pi} \sigma^2} e^{-(x-\mu)^2 / 2\sigma^2}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$\mu$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td>Pareto</td>
<td>$\text{Pa}(\alpha, \beta)$</td>
<td>$f(x) = \beta \alpha^\beta x^{\beta-1} / \Gamma(\beta)$</td>
<td>$x \in (\alpha, \infty)$</td>
<td>$\frac{\alpha \beta}{\beta-1}$</td>
<td>$\frac{\alpha^2 \beta}{(\beta-1)^2 (\beta-2)}$</td>
</tr>
<tr>
<td>Poisson</td>
<td>$\text{Po}(\lambda)$</td>
<td>$f(x) = \frac{\lambda^x e^{-\lambda}}{\Gamma(x+1)}$</td>
<td>$x \in \mathbb{Z}_+$</td>
<td>$\lambda$</td>
<td>$\lambda$</td>
</tr>
<tr>
<td>Snedecor F</td>
<td>$F(\nu_1, \nu_2)$</td>
<td>$f(x) = \frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2}) \Gamma(\frac{\nu_2}{2})} \times x^{\frac{\nu_1}{2} - 1} \left[1 + \frac{\nu_1}{\nu_2} x\right]^{-\frac{\nu_1+\nu_2}{2}}$</td>
<td>$x \in \mathbb{R}_+$</td>
<td>$\frac{\nu_1}{\nu_2-2}$</td>
<td>$\left(\frac{\nu_2}{\nu-2}\right)^2 2(\nu_1+\nu_2-2) / (\nu_1(\nu_2-4))$</td>
</tr>
<tr>
<td>Student t</td>
<td>$t(\nu)$</td>
<td>$f(x) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\left[\Gamma\left(\frac{\nu}{2}\right)\pi\right]^{1/2}} \left[1 + \frac{x^2}{\nu}\right]^{-\frac{\nu+1}{2}}$</td>
<td>$x \in \mathbb{R}$</td>
<td>$0$</td>
<td>$\nu / (\nu-2)$</td>
</tr>
<tr>
<td>Uniform</td>
<td>$\text{Un}(a, b)$</td>
<td>$f(x) = \frac{1}{b-a}$</td>
<td>$x \in (a, b)$</td>
<td>$\frac{a+b}{2}$</td>
<td>$\frac{(b-a)^2}{12}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$\text{We}(\alpha, \beta, \gamma)$</td>
<td>$f(x) = \frac{\alpha(x-\gamma)^{\alpha-1} e^{-[(x-\gamma)/\beta]^\alpha}}{\Gamma(1+\alpha^{-1})}$</td>
<td>$x \in (\gamma, \infty)$</td>
<td>$\gamma + \beta \Gamma(1+\alpha^{-1})$</td>
<td></td>
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