Sta 205 : Homework 1

Due : January 23, 2008

I. Fields and σ - fields.

(A) For a three-point outcome set \( \Omega = \{a, b, c\} \) and \( C := \{\{a\}\} \), enumerate the class \( \mathcal{A} \) of all σ-fields \( \mathcal{F} \) on \( \Omega \) that contain \( C \), i.e., satisfy \( C \subset \mathcal{F} \). Also find \( \sigma(C) \).

(B) For each integer \( n \in \mathbb{N} := \{1, 2, 3, \ldots\} \), set \( A_n := \left\{ \frac{m}{n} : m \in \mathbb{N} \right\} \).

Find \( \limsup_{n \to \infty} A_n \) and \( \liminf_{n \to \infty} A_n \).

(C) Let \( f \) and \( \{f_n\} \) be real-valued functions on any set \( \Omega \), and let \( \epsilon_k \) be any decreasing sequence such that \( \epsilon_k \downarrow 0 \) as \( k \to \infty \). Show that:

\[
\left\{ \omega : f_n(\omega) \to f(\omega) \right\} = \bigcap_{k=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \left\{ \omega : |f_n(\omega) - f(\omega)| < \epsilon_k \right\} .
\]

(D) Find a set \( \Omega \) and two fields \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) for which \( \mathcal{F}_1 \cup \mathcal{F}_2 \) is not a field.

(E) Suppose a collection \( \{\mathcal{F}_n : n \in \mathbb{N}\} \) of σ-fields satisfies the relation \( \mathcal{F}_j \subset \mathcal{F}_{j+1} \) for every \( j \in \mathbb{N} \). Does it follow that \( \bigcup \mathcal{F}_j \) is a field? (the answer is “yes” — show why)

(F) Under the same conditions, is \( \bigcup \mathcal{F}_j \) a σ-field? (this one is “no” — find a counterexample. The idea is to find a sequence \( A_n \in \mathcal{F}_n \) with \( \cup_n A_n \notin \mathcal{F}_j \) for every \( j \), hence \( \cup_n A_n \notin \bigcup_j \mathcal{F}_j \)).

(G) Let \( \Omega := (0, 1] \) be the half-open unit interval, and let \( n \in \mathbb{N} \) be a FIXED positive integer (say, three). Set

\[
\mathcal{B}_n := \{(0, j/2^n], j \in \{0, 1, \ldots, 2^n\}\},
\]

the collection of half-open intervals from zero up to an integral multiple of \( 2^{-n} \). Describe in words the σ-field

\[
\mathcal{F}_n := \sigma(\mathcal{B}_n)
\]

generated by \( \mathcal{B}_n \). Tell how many elements \( \mathcal{B}_n \) has, and how many elements \( \mathcal{F}_n \) has.
II. Asymptotic Density.

For any subset $A \subseteq \mathbb{N}$ of positive integers write $\#(A)$ for its cardinality, i.e., the number of elements it contains. The set $A$ has “asymptotic density” $d(A)$ if the limit

$$
\lim_{n \to \infty} \frac{\#(A \cap \{1, 2, \ldots, n\})}{n}
$$

exists (i.e., the lim-sup and lim-inf coincide). Denote by

$$
\mathcal{A} := \{A \subseteq \mathbb{N} : d(A) \text{ exists}\}
$$

the collection of all subsets of $\mathbb{N}$ that have an asymptotic density.

(A) Show carefully that the set of even numbers $E := 2\mathbb{N} = \{2, 4, \ldots\}$ has asymptotic density $d(E) = 1/2$. By “carefully” I mean: compute the ratio in Equation (1) exactly, for every $n$ (consider even and odd $n$ separately), and compute the limit.

(B) Find a set $B \subseteq \mathbb{N}$ that does not have an asymptotic density.

(C) Find a set $A \in \mathcal{A}$ for which $A \cap E \notin \mathcal{A}$, and conclude that $\mathcal{A}$ is not a field. Hint: the on-line lecture notes might help if you get stuck.