Final Examination

STA 205: Probability and Measure Theory

Due Saturday, 2004 May 1, 2:00 pm
(or any time before that)

This is an open-book 24-hour take-home examination. You must do your own work— no collaboration is permitted. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others (whether or not they are taking this exam). You can reach me by telephone (w: 684-3275; h: 688-0435) or, better, by e-mail (`wolpert@stat.duke.edu`).

You must **show** your **work** to get partial credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

This exam is due by 2pm Saturday, 2004 May 1. You may slip it under my office door (211c Old Chem) or hand it to me earlier.

Print Name: ____________________

Begin Time: ___ : ___ , April ___ , 2004

Finish Time: ___ : ___ , ______ , 2004

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
6.	/20
Total:	/120

Problem 1: Let $\xi_n \sim \mathsf{Ge}(1/2)$ be independent geometric-distributed random variables with probability mass function (pmf)

$$\mathsf{P}[\xi_n = x] = \begin{cases} 2^{-x} & x = 1, 2, ... \\ 0 & \text{other } x. \end{cases}$$

and set

$$X_n = 2^n \, 1_{\{n\}}(\xi_n) = \begin{cases} 2^n & \xi_n = n \\ 0 & \xi_n \neq n. \end{cases}$$

a. (5) Find the mean $\mu_n = \mathsf{E}[X_n]$ and variance $\sigma_n^2 = \mathsf{Var}[X_n]$ of X_n (*not* of ξ_n):

$\mu_n =$ ______________ $\sigma_n^2 =$ ______________

b. (5) Set $S_n \equiv X_1 + ... + X_n$. Does S_n converge as $n \to \infty$? In what sense(s)? ◯ *a.s.* ◯ *i.p.* ◯ L^1 ◯ L^2 ◯ L^∞ ◯ *vg.* Why?

c. (5) Does $X_n \to 0$ almost surely? Y N Why?

d. (5) Does $X_n \to 0$ in L^∞? Y N Why?

Problem 2: Let $(\Omega, \mathcal{F}, \mathsf{P}) \equiv \big((0,1], \mathcal{B}_1, d\omega\big)$ be the unit interval with Lebesgue measure (length). Define three random variables by

$$W(\omega) = \omega \quad X(\omega) = 1_{(0,0.6]}(\omega) \quad Y(\omega) = 1_{(0.4,1]}(\omega) \quad Z(\omega) = 1_{(0.4,0.6]}(\omega).$$

a. (15) Which (if any) of the following inclusions hold? Prove your answers.

(3) Y N $\sigma(X) \subset \sigma(Y,Z)$

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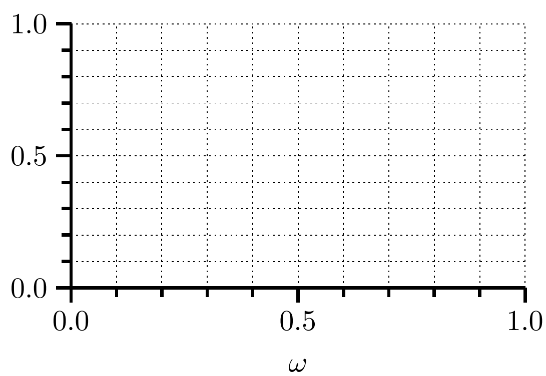
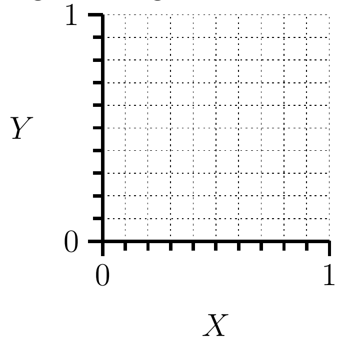
(3) Y N $\sigma(X,Y,Z) \subset \sigma(W)$

b. (5) Find and plot the conditional expectation:

$\mathsf{E}[X|Y] =$ ____________________



Problem 3: The random variables X and Y are independent with the uniform distribution on $(0, 1]$, and Z is the indicator function of the event that the point (X, Y) in the unit square lies in the "north-east" right triangle with vertices $(0, 1)$, $(1, 0)$, and $(1, 1)$ (sketch this event).



a. (5) Find the (marginal) probability distribution of Z.

b. (5) Are X and Z independent? Y N Prove it.

Problem 3: (cont'd)

c. (5) Find the conditional expectation of X, given $\sigma(Z)$:

$\mathsf{E}[X|Z] =$ ____________________

d. (5) Find the conditional expectation of Z, given $\sigma(X)$:

$\mathsf{E}[Z|X] =$ ____________________

Problem 4: Let $\Omega \equiv \{a, b, c, d, e, f, g, h, i, j\}$ consist of the indicated ten points ω and let $\mathcal{F} \equiv 2^\Omega$ be the power set of all 1024 events.

a. (10) Give a random variable $X : \Omega \to \mathbb{R}$ and a probability assignment $\mathsf{P} : \mathcal{F} \to [0, 1]$ for which X has the same probability distribution as a fair die— $\mathsf{P}[X = x] = 1/6$ for $x \in \{1, 2, 3, 4, 5, 6\}$, otherwise $\mathsf{P}[X = x] = 0$. Give each $X(\omega)$ explicitly, and give any clear complete specification of P.

Problem 4: (cont'd) Recall $\Omega = \{a, b, c, d, e, f, g, h, i, j\}$, $\mathcal{F} = 2^\Omega$, and your definitions of $X : \omega \to \mathbb{R}$ and $\mathsf{P} : \mathcal{F} \to [0, 1]$. For $n \in 1...6$ set $Y_n = 1_n(X) = \begin{cases} 1 & X = n \\ 0 & X \neq n \end{cases}$ and let $\mathcal{G} \equiv \sigma\{Y_2, Y_4, Y_6\}$ be the σ-algebra generated by the indicated random variables.

b. (5) Find $\mathsf{E}[X \mid \mathcal{G}]$.

c. (5) Find $\mathsf{E}[Y_1 \mid \mathcal{G}]$.

Problem 5: The random variables $\{U_i\}$ are independent and identically distributed, all with the uniform $\mathsf{Un}(0,1)$ distribution, and set

$$X_i \equiv (U_i)^{-2/3}.$$

Let $S_n \equiv \sum_{i=1}^{n} X_i$ be their partial sum.

a. (10) For which (if any) $p \geq 1$ is $X_i \in L^p$? Why?

b. (5) Does the central limit theorem apply to S_n? Y N
If so, find sequences $\{a_n\} \subset \mathbb{R}$ and $\{b_n\} \subset \mathbb{R}_+$ for which

$$(S_n - a_n)/b_n \Rightarrow \mathsf{No}(0,1);$$

$a_n =$ ______________, $b_n =$ ______________;
if not, explain why it is not possible.

Problem 5: (cont'd) Recall $U_i \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$, $X_i \equiv (U_i)^{-2/3}$, and $S_n \equiv \sum_{i=1}^n X_i$.

c. (5) Does the Law of Large Numbers apply to S_n? Y N
If so, find the limit

$$\lim_{n\to\infty} \frac{1}{n} S_n = \underline{\hspace{4cm}};$$

if not, explain why.

d. (xc) Find the approximate distribution of S_{1000}. Explain.

Problem 6: Let X_n be a sequence of real-valued random variables on some fixed probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and define

$$Y_n \equiv \sin X_n.$$

a. (5) Can you find distributions for X_n so that $||Y_n||_2 \to \infty$ as $n \to \infty$? Y N If so, do it; if not, say why.

b. (5) If $X_n \to 0$ in probability, does it follow that $Y_n \to 0$ in probability? Y N Why, or why not?

c. (5) If $X_n \to 0$ in probability, does it follow that $Y_n \to 0$ in L^2? Y N Why, or why not?

d. (5) If $X_n \to 0$ *in distribution*, in which (if any) way(s) does it follow that $Y_n \to 0$? Give a *brief* explanation.

◯ *a.s.* ◯ *i.p.* ◯ L^1 ◯ L^2 ◯ L^∞ ◯ *vg.*