Final Examination

STA 205: Probability and Measure Theory

Due by Monday, 2005 May 2, 12:00 n

This is an open-book 24-hour take-home examination. You must do your own work— no collaboration is permitted. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others (whether or not they are taking this exam). You can reach me by telephone (w: 684-3275; h: 688-0435) or, better, by e-mail (`wolpert@stat.duke.edu`).

You must **show** your **work** to get partial credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

This exam is due by 12n Monday, 2005 May 2. You may slip it under my office door (211c Old Chem) or hand it to me earlier.

Print Name: ______________________

Issued: 2:50 , April 27 , 2005

Unsealed: ___ : ___ , April ___ , 2005

Finished: ___ : ___ , ________ , 2005

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
6.	/20
Total:	/120

Problem 1: Let $Z \sim \mathsf{Ex}(2)$ have an exponential distribution with mean $\mathsf{E}Z = 1/2$; set $X \equiv \lfloor Z \rfloor$, the integer part of Z, and $Y \equiv Z - X$, the fractional part of Z.

a. Find the probability distributions for X and Y. For each, give the p.d.f. or p.m.f. (at every point) and give the means $\mathsf{E}X$ and $\mathsf{E}Y$.

b. Are X and Y independent? Prove or disprove independence.

Problem 1 (cont'd): Recall $Z \sim \mathsf{Ex}(2)$, $X \equiv \lfloor Z \rfloor$, and $Y \equiv Z - X$.

c. Find the indicated conditional expectations:

$\mathsf{E}[Z \mid X] =$ __________

$\mathsf{E}[Z \mid Y] =$ __________

$\mathsf{E}[Z \mid X, Y] =$ __________

Problem 2: Let $X \sim \mathsf{Bi}(n, \frac{2}{n})$ have the Binomial distribution with $n \in \mathbb{N}$ and $p = 2/n$; let $Y \sim \mathsf{Po}(\lambda)$ have a Poisson distribution with mean $\lambda > 0$; and set $Z \equiv (Y - \lambda)/\sqrt{\lambda}$.

a. Find the characteristic functions $\phi_X(\omega) = \mathsf{E}e^{i\omega X}$, $\phi_Y(\omega) = \mathsf{E}e^{i\omega Y}$, and $\phi_Z(\omega) = \mathsf{E}e^{i\omega Z}$. Show your work.

$\phi_X(\omega) \quad = \quad$ ____________________

$\phi_Y(\omega) \quad = \quad$ ____________________

$\phi_Z(\omega) \quad = \quad$ ____________________

Problem 2 (cont'd): Recall $X \sim \mathsf{Bi}(n, \frac{2}{n})$, $Y \sim \mathsf{Po}(\lambda)$, and $Z \equiv (Y-\lambda)/\sqrt{\lambda}$.

b. Find the limit in distribution for X as $n \to \infty$. [Hint: $(1+x/n)^n \to$???]

c. Find the limit in distribution for Z as $\lambda \to \infty$. [Hint: $\exp(\epsilon\, x) = 1 + \epsilon\, x + \epsilon^2 x^2/2 + o(\epsilon^2)$ as $\epsilon \to 0$].

Problem 3: In each problem below, $\{X_j\}$ are independent and identically distributed, with the distribution indicated in each problem. Circle True or False to indicate whether or not the indicated sequence of random variables converges to zero as indicated. Justify your answer.

a. T F $\frac{1}{n}S_n \to 0$ *a.s.*, where $S_n = \sum_{j \le n} X_j$ and $\mathsf{P}[X_j = \pm 1] = \frac{1}{2}$.

b. T F $Y_n \to 0$ *a.s.*, where $Y_n = n 1_{[n \cdot X_n < 1]}$ and $X_j \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$.

c. T F $Z_n \to 0$ *a.s.*, where $Z_n = 2^n 1_{[X_n < 2^{-n}]}$ and $X_n \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$.

d. T F $\frac{1}{n}S_n \to 0$ *a.s.*, where $S_n = \sum_{j \le n} X_j^{-1}$ and $X_j \overset{\text{iid}}{\sim} \mathsf{No}(0,1)$.

Problem 4: Let $\{X_n > 0\}$ and $X > 0$ be positive random variables with $X_n \to X$ *a.s.* Choose True or False below, and give a proof (*i.e.*, cite a relevent theorem) or counter-example to show that you're right.

a. T F $\frac{1}{X_n} \to \frac{1}{X}$ almost surely.

b. T F $X_n \to X$ in L_1 if each $X_n \le \pi$.

c. T F $\mathsf{E}\left[\frac{1}{X_n}\right] \to \mathsf{E}\left[\frac{1}{X}\right]$ if each $X_n \le \pi$.

Problem 4 (cont'd): Recall $\{X_n > 0\}$, $X > 0$, and $X_n \to X$ *a.s.*

d. T F $\log(X_n) \to \log(X)$ in probability.

e. T F $\mathsf{E}\log(X) \le \log(\mathsf{E}X)$ if $(X + X^{-1}) \in L_1$.

f. T F $\liminf \mathsf{E}|\log(X_n)| \ge \mathsf{E}|\log(X)|$

Problem 5: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, and $X : \Omega \to \mathbb{R}$ a real-valued random variable (in particular, $|X(\omega)| < \infty$ for every $\omega \in \Omega$). For each question below, give a proof or counter-example.

a. Is it possible to have $\mathsf{E}|X| = \infty$, if Ω is finite?

b. Is it possible to have $\mathsf{E}|X| = \infty$, if Ω is countably infinite?

c. If X has a probability density function $f(x)$, is it possible for the random variable $Y \equiv f(X)$ to satisfy $Y \notin L_1$?

d. Find a sequence X_n of random variables that converge *a.s.* to X, and satisfy $\mathsf{E}|X_n - X| \to 0$, but that do *not* converge to X in L_1 [Hint: make $\mathsf{E}|X_n| = \infty$].

Problem 6: Let $\xi_j \sim \mathsf{Bi}(1,p)$ be independent Bernoulli random variables taking the values one and zero with probabilities p and $1-p$, and let $\alpha, \beta \in \mathbb{R}$ be real numbers. Set $S_n \equiv \sum_{j=1}^{n} \xi_j$ and:

$$X_n \equiv \alpha S_n - \beta n \qquad Y_n \equiv e^{\alpha S_n - \beta n} \qquad Z_n \equiv (S_n - \beta n)^2 - \alpha n$$

a. For which (if any) $\alpha, \beta \in \mathbb{R}$ is X_n a martingale? For which (if any) of these is it also U.I.?

b. For which (if any) $\alpha, \beta \in \mathbb{R}$ is Y_n a martingale? For which (if any) of these is it also U.I.?

c. For which (if any) $\alpha, \beta \in \mathbb{R}$ is Z_n a martingale? For which (if any) of these is it also U.I.?

Name	**Notation**	**pdf/pmf**	**Range**	**Mean μ**	**Variance σ^2**	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,\ldots\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,\ldots\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\beta)$	$f(x)=\beta\,\alpha^\beta/x^{\beta+1}$	$x\in(\alpha,\infty)$	$\frac{\alpha\,\beta}{\beta-1}$	$\frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta,\gamma)$	$f(x)=\frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha}e^{-[(x-\gamma)/\beta]^\alpha}$	$x\in(\gamma,\infty)$	$\gamma+\beta\Gamma(1+\alpha^{-1})$		