Midterm Examination

STA 205: Probability and Measure Theory

Wednesday, 2005 Mar 9, 2:50-4:05 pm

This is a closed-book examination. You may use a single one-sided sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others.

Unless a problem states otherwise, you must **show** your **work** to get partial credit. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

Print Name: ______________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the probability space $\Omega = \{a, b, c, d, e\}$ with just five points and sigma-field $\mathcal{F} = 2^\Omega$.

a. (6) Define random variables X and Y by

$$X(\omega) = \begin{cases} 1 & \omega \in \{a, b\} \\ 0 & \omega \notin \{a, b\} \end{cases} \qquad Y(\omega) = \begin{cases} 2 & \omega \in \{a, c\} \\ 0 & \omega \notin \{a, c\} \end{cases}.$$

Give *explicitly* (by listing all elements) the σ-fields $\mathcal{F}_X$ and $\mathcal{F}_Y$ generated by X and Y, respectively.

$$\mathcal{F}_X = \Big\{ \qquad\qquad\qquad\qquad \Big\}$$
$$\mathcal{F}_Y = \Big\{ \qquad\qquad\qquad\qquad \Big\}$$

b. (4) Find the σ-field generated by X and Y, $\mathcal{F}_{X,Y} = \mathcal{F}_X \vee \mathcal{F}_Y$:

$$\mathcal{F}_{X,Y} = \Bigg\{ \qquad\qquad\qquad\qquad \Bigg\}$$

Is this the same as $\sigma(Z)$, for $Z \equiv X + Y$? ◯ Yes ◯ No

c. (5) For numbers $\alpha, \beta \geq 0$ with $\alpha + \beta \leq 1/2$, let P be the probability measure on $\mathcal{F}$ determined by the relations

$$\mathsf{P}(\{a\}) = \mathsf{P}(\{b\}) = \alpha \quad \mathsf{P}(\{c\}) = \mathsf{P}(\{d\}) = \beta \quad \mathsf{P}(\{e\}) = 1 - 2(\alpha + \beta).$$

Find all α, β for which $\mathcal{F}_X$ and $\mathcal{F}_Y$ are independent. Simplify!

d. (5) Find all α, β for which X and $Z \equiv X + Y$ are independent.

Problem 2: For $n \in \mathbb{N}$ let $Z_n \sim \mathsf{Ex}(1)$ be independent random variables, all with the exponential distribution with density function

$$f(z) = e^{-z}\, 1_{\mathbb{R}_+}(z).$$

a. (4) True or false? Circle one and prove your answer. T F

$$\limsup_{n\to\infty} Z_n = \infty$$

b. (4) True or false? Circle one and prove your answer. T F

$$\liminf_{n\to\infty} Z_n = -\infty$$

c. (4) True or false? Circle one and prove your answer. T F

$$\mathsf{P}[Z_n > n \ \text{ infinitely often}] = 1$$

d. (4) Evaluate the indicated limit:

$$\lim_{n\to\infty} \mathsf{P}\left[\min_{1\le k\le n} Z_k \ge \frac{1}{n}\right] = \underline{\hspace{6cm}}$$

e. (4) Is the event $\left[\min_{1\le k\le n} Z_k \ge \frac{1}{n}\right]$ from d. above in the tail σ-field $\mathcal{T} = \cap_n \sigma\{Z_m : m \ge n\}$? Why or why not? Y N

Problem 3:

Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables, all defined on the same probability space $(\Omega, \mathcal{F}, \mathsf{P})$. The event that the infimum (greatest lower bound) is at least $\alpha \in \mathbb{R}$ can be expressed $\cap_{n=1}^{\infty}[X_n \geq \alpha]$.

a. (8) Write the event that the supremum of $\{X_n\}_{n\in\mathbb{N}}$ is less than or equal to $\beta \in \mathbb{R}$, using intersections, unions, etc. to combine events of the form $[X_n \leq x]$, $[X_n \geq y]$, etc.:

$[\sup_{n<\infty} X_n \leq \beta] =$ ______________________________

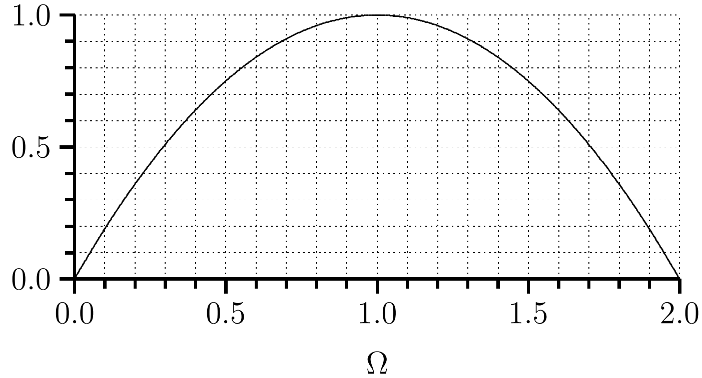
b. (8) Write the event that only finitely-many X_n's exceed β (Hint: Think about $\{X_n : n \geq m\}$ for $m \in \mathbb{N}$):

$[\limsup_{n\to\infty} X_n \leq \beta] =$ ______________________________

c. (4) Why does this show that the lim-sup $X = \limsup_{n\to\infty} X_n$ of any sequence of $\mathcal{G}$-measureable random variables is a $\mathcal{G}$-measureable random variable, for any σ-field $\mathcal{G} \subset \mathcal{F}$?

Problem 4: Let $X = \omega(2-\omega)$ be a random variable on the space $\Omega = (0,2]$ with $\mathcal{F} = \mathcal{B}(\Omega)$, the Borel sets (it's plotted below).

a. (10) Find *and plot* a non-negative **simple** random variable $Y \in \mathcal{E}_+$ satisfying $Y(\omega) \le X(\omega)$ and $Y(\omega) \ge X(\omega) - 0.5$ for all $\omega \in \Omega$.



$Y(\omega) =$

b. (6) For $c > 0$ define an additive set function P on $\mathcal{F}$ by

$$\mathsf{P}\{(a,b]\} = c^{-1} \int_a^b 2\,\omega\, d\omega = (b^2 - a^2)/c$$

for $(a,b] \subset \Omega$. For which c is P a probability measure on $(\Omega, \mathcal{F})$? Find $\mathsf{E}X$ and $\mathsf{E}Y$ for this probability measure.

$c =$ ________ $\mathsf{E}X =$ __________ $\mathsf{E}Y =$ __________

c. (4) Let $Z = 1_{(0,1]}$. Are X and Z independent on $(\Omega, \mathcal{F}, \mathsf{P})$? Why?
◯ Yes ◯ No

Problem 5: Let $(\Omega, \mathcal{F}, \mathsf{P})$ be the natural numbers $\Omega = \mathbb{N} = \{1, 2, 3, \dots\}$ with $\mathcal{F} = 2^\Omega$ and $\mathsf{P}[A] = \sum\{2^{-\omega} : \ \omega \in A \cap \mathbb{N}\}$.

a. (5) Is the random variable $X(\omega) = \omega$ in $L^1(\Omega, \mathcal{F}, \mathsf{P})$? If so, find $\mathsf{E}[X]$; if not, tell why. ◯ Yes ◯ No Reasoning:

b. (5) For $n \in \mathbb{N}$ define a random variable Y_n by $Y_n(\omega) = 0$ if $\omega < n$, $Y_n(\omega) = 2^n$ if $\omega \geq n$. Does the Monotone Convergence Theorem apply to $\{Y_n\}$? If so, tell what MCT says; if not, explain.
◯ Yes ◯ No Reasoning:

c. (5) Define Y_n as above. Does Fatou's Lemma apply? If so, verify Fatou's conclusion by calculation; if not, why? ◯ Yes ◯ No
Reasoning:

d. (5) If possible, construct independent random variables Z_1, Z_2 on $(\Omega, \mathcal{F}, \mathsf{P})$, with (non-trivial) Bernoulli $\mathsf{Bi}(1, p_j)$ distributions, *i.e.*, with $\mathsf{P}[Z_j = 1] = p_j = 1 - \mathsf{P}[Z_j = 0]$, for some $p_1, p_2 \in (0, 1)$. Is it possible to have $p_1 = p_2 = 1/2$? ◯ Yes ◯ No Explain.

Blank Worksheet

Another Blank Worksheet