Final Examination

STA 205: Probability and Measure Theory

Due by Friday, 2006 May 5, 7:00 pm

This is an open-book take-home examination. You may work on it during any consecutive 24-hour period you like; please record your starting and ending times on the lines below.

You must do your own work— no collaboration is permitted. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others (whether or not they are taking this exam). You can reach me by telephone (w: 684-3275; h: 688-0435) or, better, by e-mail (`wolpert@stat.duke.edu`).

You must **show** your **work** to get partial credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

This exam is due by 7pm Friday, 2006 May 5. You may slip it under my office door (211c Old Chem) or hand it to me earlier.

Print Name:	______________
Issued:	4:05 , Apr 24 , 2006
Unsealed:	: , , 2006
Finished:	: , , 2006
Due by:	7:00 , May 5 , 2006

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Problem 1: Let $\xi_j \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ be standard (unit-mean) exponentially-distributed random variables; set

$$X_n \equiv \max_{j \le n} \xi_j.$$

a) Find the distribution function for X_n, correct for *all* $x \in \mathbb{R}$:

$F_n(x) = \Bigg\{$

b) Find a sequence $a_n \to \infty$ of (non-random) real numbers for which $(X_n - a_n)$ has a non-trivial limiting distribution, and find the distribution function for the limit, correctly for all $z \in \mathbb{R}$:

$(X_n - a_n) \Rightarrow Z$, $a_n =$ ______________,

$F_Z(z) \equiv \mathsf{P}[Z \le z] =$

Problem 1 (cont'd): Recall that $\xi_j \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$, and set

$$Y_n \equiv \min_{j \le n} \xi_j.$$

c) Find the distribution function for Y_n, correct for *all* $y \in \mathbb{R}$:

$$F_n(y) = \begin{cases} \\ \\ \end{cases}$$

d) Find a sequence $b_n \to \infty$ of positive real numbers for which $b_n \cdot Y_n$ has a non-trivial limiting distribution, and find the distribution function for the limit (correctly for all $z \in \mathbb{R}$):

$b_n \cdot Y_n \Rightarrow Z$, $b_n =$ ______________,

$F_Z(z) \equiv \mathsf{P}[Z \le z] =$

Problem 2: Fix $p \in \mathbb{N}$ and let $\Omega \equiv \{1, \cdots, p\}$, with power set $\mathcal{F} \equiv 2^\Omega$, and let $\mathsf{P}(A) \equiv |A|/p$ for $A \in \mathcal{F}$ be the uniform probability measure on Ω.

a) If p is prime, and if $A, B \in \mathcal{F}$ are independent (*i.e.*, $A \perp\!\!\!\perp B$), show that either A or B is Ω or $\emptyset$.

b) Find the joint characteristic function

$$\phi(t_1, t_2) = \mathsf{E}[e^{it_1 X_1 + it_2 X_2}]$$

for the indicator random variables $X_i(\omega) \equiv 1_i(\omega)$. Use it to confirm that X_1 and X_2 are not independent.

Problem 2 (cont'd): Recall that $\Omega \equiv \{1, \cdots, p\}$, $\mathcal{F} \equiv 2^\Omega$, and $\mathsf{P}(A) \equiv |A|/p$ for $A \in \mathcal{F}$. Now fix any $n \in \mathbb{N}$ and let $\Omega^n = \{\vec{\omega} = (\omega_1, \cdots, \omega_n)\}$ be the Cartesian product, with product σ-field $\mathcal{F}^n$ and product measure $\mathsf{P}^n = \mathsf{P} \otimes \mathsf{P} \otimes \cdots \otimes \mathsf{P}$, describing n independent uniform draws from Ω. For $1 \le i \le j \le n$ consider the $\binom{n}{2}$ events

$$A_{ij} \equiv \{\vec{\omega} \in \Omega^n : \ \omega_i = \omega_j\}$$

c) Find $\mathsf{P}[A_{ij}]$ correctly for *each* of the pairs $1 \le i \le j \le n$.

d) Are the $\binom{n}{2}$ events $\{A_{ij}\}$ pairwise-independent? Prove it.

e) Are the $\binom{n}{2}$ events $\{A_{ij}\}$ independent? Prove it.

Problem 3: Let $X, Y \overset{\text{iid}}{\sim} \mathsf{Ex}(\lambda)$ be independent random variables, each with the exponential distribution with mean $1/\lambda$. Find:

a) The probability:

$\mathsf{P}[X \geq 2Y] =$ ____________________

b) The p.d.f. for $Z \equiv Y/X$, correctly for all $z \in \mathbb{R}$:

$f_Z(z) =$ ____________________

Problem 3 (cont'd): Recall $X, Y \overset{\text{iid}}{\sim} \mathsf{Ex}(\lambda)$ and $Z \equiv Y/X$. Find:

c) The indicated expectation:

$\mathsf{E}[Z] =$ ____________________

d) Show that the characteristic function $\phi(t) \equiv \mathsf{E}[e^{it\log(Z)}]$ for $\log(Z)$ is an even real-valued function, *i.e.*, satisfies $\phi(-t) = \phi(t) = \overline{\phi(t)}$
(XC: Find $\phi(t)$ explicitly):

Problem 4: Let $\{X_n > 0\}$ and $X > 0$ be positive random variables with $X_n \to X$ *a.s.* Choose True or False below, and sketch a proof (*e.g.*, cite a relevant theorem) or give a counter-example to show that you're right.

a) T F $\log(X_n) \to \log(X)$ in probability.

b) T F $X_n \to X$ in L_2 if each $\mathsf{E}\left[|X_n|^3\right] \le \pi$.

c) T F $\log(X_n) \to \log(X)$ in L_1 if each $\mathsf{E}[|X_n|^3] \le \pi$.

Problem 4 (cont'd): Recall $\{X_n > 0\}$, $X > 0$, and $X_n \to X$ *a.s.*

d) T F $\limsup_{n\to\infty} \mathsf{E}[\log(1 + X_n)] \geq \mathsf{E}[\log(1 + X)]$

e) T F $X \in L_2$ if $\mathsf{E}[\exp(t \cdot X)] < \infty$ for some $t > 0$.

f) T F $X \in L_2$ if $\mathsf{E}[\exp(t \cdot X)] < \infty$ for some $t < 0$.

Problem 5: Let $(\Omega, \mathcal{F}, \mathsf{P}) = \big((0,1], \mathcal{B}_1, \lambda\big)$ be the unit interval with Lebesgue measure $\lambda(d\omega) = d\omega$ (length). Define three random variables by

$$X(\omega) \equiv 1_{(0,.5]}(\omega) \qquad Y(\omega) \equiv 1_{(.3,0.7]}(\omega) \qquad Z(\omega) \equiv \omega(1-\omega).$$

a) For each pair of σ-algebras below, indicate with a symbol on the line whether there is inclusion ($\subset$ or $\supset$), equality ($=$), independence ($\perp\!\!\!\perp$), or none of these (X):

$\sigma(X)$ ____ $\sigma(Y)$

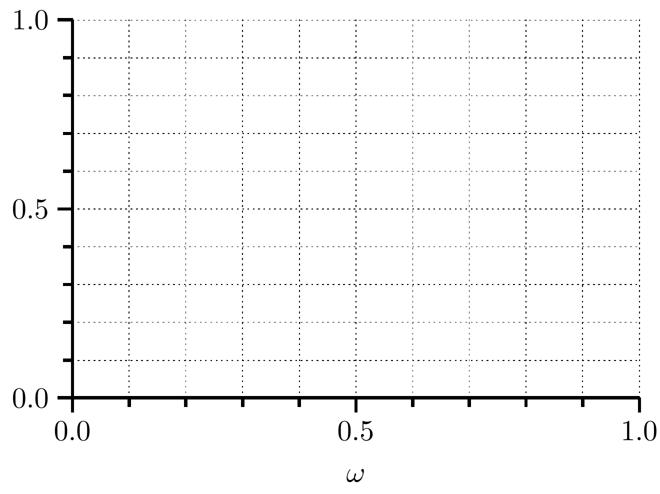
$\sigma(X)$ ____ $\sigma(Z)$

$\sigma(Y)$ ____ $\sigma(Z)$

$\sigma(X,Y)$ ____ $\sigma(Z)$

$\sigma(X,Z)$ ____ $\mathcal{F}$

b) Find and plot the conditional expectation $\mathsf{E}[Z|X,Y]$:

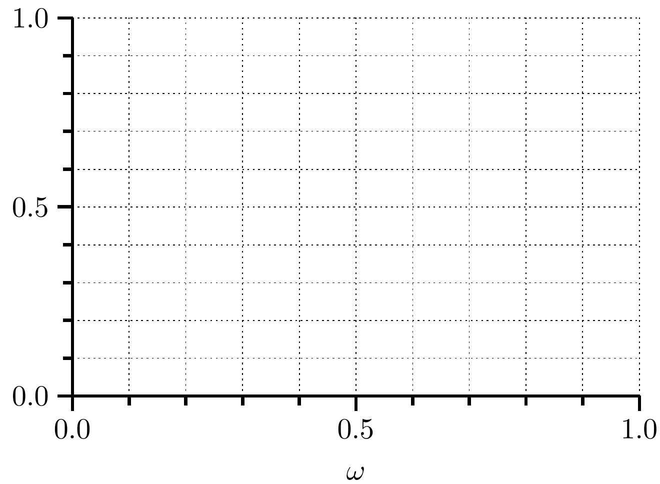


Problem 5 (cont'd): Recall $(\Omega, \mathcal{F}, \mathsf{P}) = \big((0,1], \mathcal{B}_1, \lambda\big)$ and

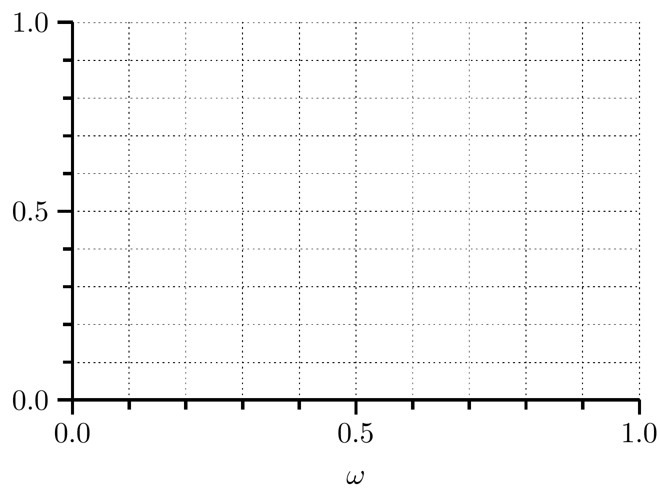
$$X(\omega) \equiv 1_{(0,.5]}(\omega) \qquad Y(\omega) \equiv 1_{(.3,0.7]}(\omega) \qquad Z(\omega) \equiv \omega(1-\omega).$$

Define a new random variable by $W(\omega) \equiv 1_{(0.2,0.6]}(\omega)$

c) Find and plot the conditional expectation $\mathsf{E}[W|Y]$:



d) Find and plot the conditional expectation $\mathsf{E}[W|Z]$:



Problem 6: Let $\xi_j \sim \mathsf{Un}(0,2)$ be independent random variables with the uniform distribution on the interval $(0,2)$, and set $X_n \equiv \prod_{j=1}^n \xi_j$.

a) Find the mean and variance of X_n:

$\mathsf{E}[X_n] =$ __________________ $\mathsf{V}[X_n] =$ __________________

b) For each $p > 0$, can you find a number $a_p > 0$ such that

$$M_p(n) \equiv (a_p)^n \cdot (X_n)^p$$

is a martingale?

c) Bound as tightly as you can

$$\mathsf{P}\left[X_{16} > 0.010\right] \leq \underline{\hspace{4cm}}$$

d) Bound as tightly as you can

$$\mathsf{P}\left[\sup_{0 \leq n \leq 16} X_n > 2.0\right] \leq \underline{\hspace{4cm}}$$

e) (XC) For $r, \lambda > 0$ find bounds for $\mathsf{P}[\sup_{n<\infty} r^n X_n > \lambda]$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\beta)$	$f(x)=\beta\,\alpha^\beta/x^{\beta+1}$	$x\in(\alpha,\infty)$	$\frac{\alpha\,\beta}{\beta-1}$	$\frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta,\gamma)$	$f(x)=\frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha}e^{-[(x-\gamma)/\beta]^\alpha}$	$x\in(\gamma,\infty)$	$\gamma+\beta\Gamma(1+\alpha^{-1})$		