Midterm Examination

STA 205: Probability and Measure Theory

Wednesday, 2006 Mar 8, 2:50-4:05 pm

This is a closed-book examination. You may use a single one-sided sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others.

Unless a problem states otherwise, you must **show** your **work** to get partial credit. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

Print Name: ____________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1. For $n \in \mathbb{N}$ let $Z_n \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ be independent random variables, all with the exponential distribution with density function

$$f(z) = e^{-z}\, 1_{\mathbb{R}_+}(z).$$

Give short answers below, **with** a brief **justification**:

a. (4) Is $\{Z_n\}$ uniformly integrable (U.I.)? ◯ Yes ◯ No
Why?

b. (4) $\mathsf{P}\big[\{Z_n > \log(n)\} \text{ for infinitely-many } n\ \big] =$ ☐
Why?

c. (4) $\mathsf{P}\big[\{Z_n > 2\log(n)\} \text{ for infinitely-many } n\ \big] =$ ☐
Why?

d. (4) Set $X_n \equiv \big\{\min\limits_{1\le k\le n} Z_k\big\}$; find $\lim_{n\to\infty} \mathsf{P}[X_n > 2/n] =$ ☐
Why?

e. (4) Set $Y_n \equiv \{\max\limits_{1\le k\le n} Z_k\}$; find $\lim_{n\to\infty} \mathsf{P}\big[Y_n \le \log(n)\big] =$ ☐
Why?

Problem 2. Let $\mathcal{C} = \{C_j\}$ be the collection of subsets $C_j \equiv (0, j/4]$ of $\Omega = (0,1]$, for $j = 0, 1, 2, 3, 4$.

a. (5) Describe in detail the smallest π-system $\pi(\mathcal{C})$ containing $\mathcal{C}$. How many elements does $\pi(\mathcal{C})$ include? Describe them, don't list them.
$\#(\pi(\mathcal{C})) =$ ☐ $\pi(\mathcal{C}) =$

b. (5) Describe in detail the smallest λ-system $\lambda(\mathcal{C})$ containing $\mathcal{C}$. How many elements does $\lambda(\mathcal{C})$ include? Describe them, don't list them.
$\#(\lambda(\mathcal{C})) =$ ☐ $\lambda(\mathcal{C}) =$

c. (5) Is $\mathcal{C}$ a partition? If so, explain why; if not, describe in detail the partition $\mathcal{P}$ it determines. How many elements does $\mathcal{P}$ include?
$\#(\mathcal{P}) =$ ☐ $\mathcal{P} =$

d. (5) For $\mathsf{P}(d\omega) = d\omega$ (Lebesgue measure), give a non-trivial (*i.e.*, not a.s. constant) random variable X for which $\sigma(\mathcal{C}) \perp\!\!\!\perp \sigma(X)$.

$$X(\omega) = \begin{cases} \\ \\ \end{cases}$$

Problem 3. Let $\Omega = \mathbb{R}_+$ with the Borel sets $\mathcal{F}$ and, for some fixed $\lambda > 0$, define a probability measure on $(\Omega, \mathcal{F})$ by $\mathsf{P}(d\omega) \equiv \lambda e^{-\lambda\omega}\, d\omega$, *i.e.*,

$$\mathsf{P}\big\{(a,\ b]\big\} = e^{-\lambda a} - e^{-\lambda b}, \qquad (a,\ b] \subset \mathbb{R}_+.$$

For $n \in \{0, 1, 2, ...\}$ define random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ by

$$X_n(\omega) \equiv \omega^n / n! \qquad S_n(\omega) \equiv \sum_{k=0}^{n} X_k(\omega)$$

a. (5) For which (if any) $\lambda > 0$ does the **monotone convergence theorem** apply to the sequence $\{S_n\}$? What does the M.C.T. say?

b. (5) For which (if any) $\lambda > 0$ does the **dominated convergence theorem** apply to the sequence $\{S_n\}$? What L_1 R.V. Y dominates? What does D.C.T. say?

Problem 3 (cont).

Recall $\mathsf{P}(d\omega) = \lambda e^{-\lambda\omega}\, d\omega$, $X_n(\omega) \equiv \frac{\omega^n}{n!}$, and $S_n(\omega) \equiv \sum_{k=0}^{n} X_k(\omega)$ on $\mathbb{R}_+$.

c. (5) For which (if any) $\lambda > 0$ is the random variable X_1 independent of the random variable $Z(\omega) \equiv 1_{(0,1]}(\omega)$?

d. (5) Find the indicated expectations and limits. You may find it useful to recall that $\int_0^\infty x^n e^{-x}\, dx = \Gamma(n+1) = n!$ for $n \in \mathbb{N}$.

$\mathsf{E}[X_n] =$ ______________ $\to$ $\lim_{n\to\infty} \mathsf{E}[X_n] =$ ______________

$\mathsf{E}[S_n] =$ ______________ $\to$ $\lim_{n\to\infty} \mathsf{E}[S_n] =$ ______________

Problem 4. For 2pt each, write your answers in the boxes provided, or circle True or False. No explanations are required. In parts b) and c), $g = g(x)$ denotes a completely arbitrary Borel measureable function.

a) If $X \in L_1$ then $X \in L_2$. T F

b) If X is simple[1] and $Y = g(X)$ then Y is simple. T F

c) If X is continuous[2] and $Y = g(X)$ then Y is continuous. T F

d) If $|X_n| \le Y$ and $Y \in L_1$ then $\{X_n\}$ is U.I. T F

e) If $X \in L_2$ and $Y \in L_2$ then for what (if any) p is $X \cdot Y \in L_p$? [ ]

f) If $X \in L_1$ then $\mathsf{E}\big[\exp(X)\big] \le \exp\big(\mathsf{E}[X]\big)$. T F

g) If $X \in L_1$ and $Y_n \equiv 1_{\{|X|>n\}}$ then $\mathsf{E}[X\,Y_n] \to 0$ as $n \to \infty$. T F

h) If $\mathsf{E}[e^X] = 10$, give an upper bound for $\mathsf{P}[X > 5]$: [ ]

i) If $\mathsf{E}[e^{|X|}] < \infty$ then $X \in L_2$. T F

j) If $X \perp\!\!\!\perp Y$ and $Y \perp\!\!\!\perp Z$ then $X \perp\!\!\!\perp Z$. T F

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[1] A *simple* R.V. is one that takes on only finitely-many different values.

[2] This is a sloppy but common way of saying that the random variable has a continuous distribution, *i.e.*, that its C.D.F. $F(x)$ is a continuous function of $x \in \mathbb{R}$.

Problem 5. The Characteristic Function ("ch.f.") of any random variable Z is defined by the expectation

$$\phi_Z(s) \equiv \mathsf{E}\left[e^{isZ}\right] \qquad s \in \mathbb{R},$$

a complex-valued function of the real-valued argument s.

a. (4) If X is any random variable with ch.f. $\phi_X(s)$, and if $Y = a + b\,X$ for real numbers a and b, express the ch.f. $\phi_Y(s)$ of Y in terms of $\phi_X(s)$:

   $\phi_Y(s) =$

b. (6) Let $\xi \sim \mathsf{Un}\big([-\pi, \pi)\big)$ be uniformly distributed on the indicated interval. Find the ch.f. for ξ and for the random variable $X \equiv 0 \vee \xi$, the maximum of zero and ξ. Simplify!

   $\phi_\xi(s) =$ ______________ $\phi_X(s) =$ ______________

Problem 5 (cont).

The ch.f. of any random variable Z is still defined by

$$\phi_Z(s) \equiv \mathsf{E}\left[e^{isZ}\right] \qquad s \in \mathbb{R}.$$

c. (5) On the probability space $(\Omega, \mathcal{F}, \mathsf{P})$ with $\Omega = (0, \pi]$ with the Borel sets $\mathcal{F}$ and probability measure $\mathsf{P}(d\omega) = \frac{1}{2}\sin(\omega)\, d\omega$ proportional to the sine[3] function, find the ch.f. of the random variable $Y(\omega) = \omega$.

$\phi_Y(s) =$

d. (5) Describe all probability distributions whose ch.f. satisfies $\phi(\pi) = 1$.

[3]Hint: Since $\exp(i\omega) = \cos(\omega) + i\sin(\omega)$, then $\sin(\omega) = [e^{-i\omega} - e^{i\omega}](i/2)$.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\beta)$	$f(x)=\beta\,\alpha^\beta/x^{\beta+1}$	$x\in(\alpha,\infty)$	$\frac{\alpha\,\beta}{\beta-1}$	$\frac{\alpha^2\beta}{(\beta-1)^2(\beta-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta,\gamma)$	$f(x)=\frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha}e^{-[(x-\gamma)/\beta]^\alpha}$	$x\in(\gamma,\infty)$	$\gamma+\beta\Gamma(1+\alpha^{-1})$		