Midterm Examination

STA 205: Probability and Measure Theory

Wednesday, 2007 Mar 7, 2:50-4:05 pm

This is a closed-book examination. You may use a single sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss questions with others.

Unless a problem states otherwise, you must **show** your **work** to get partial credit. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible. Good luck.

Print Name: ____________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1. Let X, Y be independent random variables with the same charactristic function (ch.f.)

$$\phi(\omega) = \mathsf{E}e^{i\omega X} = \mathsf{E}e^{i\omega Y}.$$

a. (5) For real numbers $a, b \in \mathbb{R}$ find the ch.f. for the random variable $Z := a + b(X - Y)$:
$\phi_Z(\omega) =$

b. (5) Give lower and upper bounds for the probability $p := \mathsf{P}[X \le Y]$:

____________ $\le p \le$ ____________

c. (10) If $\phi(\omega) = \cos(\omega)$, find the probability distribution of X. What is $\mathsf{P}[X \le Y]$ for this ch.f.? [Hint: write $\cos(\omega)$ in terms of $\exp(\pm i\omega)$]

Problem 2. For $n \in \mathbb{N}$ let $U_n \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$ be independent uniformly-distributed random variables, and set $Z_n := 1/U_n$.

Give short answers below (in the boxes), **with** a brief **justification**:

a. (4) Is $\{Z_n\}$ uniformly integrable (U.I.)? ◯ Yes ◯ No
Why?

b. (4) $\mathsf{P}\big[\{Z_n > n\}$ for infinitely-many $n\ \big] =$ $\boxed{}$
Why?

c. (4) $\mathsf{P}\big[\{Z_n > 2^n\}$ for infinitely-many $n\ \big] =$ $\boxed{}$
Why?

d. (4) Set $X_n := \big\{\min_{1\le k\le n} Z_k\big\}$; find $\lim_{n\to\infty} \mathsf{P}[X_n > 1 + 1/n] =$ $\boxed{}$
Why?

e. (4) Set $Y_n := \{\max_{1\le k\le n} Z_k\}$; find $\lim_{n\to\infty} \mathsf{P}\big[Y_n \le n\big] =$ $\boxed{}$
Why?

Problem 3. Let X and Y be the coordinates of a point drawn uniformly from the disk $\Omega := \{(x, y) : x^2 + y^2 \le 1\}$ (with P given by $\frac{1}{\pi}$ times "area", or two-dimensional Lebesgue measure). Set $R := \left(X^2 + Y^2\right)^{1/2}$, the Euclidean distance from (X, Y) to $(0, 0)$. (It may be helpful to draw a picture).

a. (5) Are X and Y independent? ◯ Yes ◯ No Justify your answer. (Hint: consider the event $[X > \frac{\sqrt{2}}{2}]$).

b. (5) Is the collection of events $\mathcal{C} = \left\{A_{rx} = \{\omega : R(\omega) \le r,\ X(\omega) \le x\}\right\}$ for $r \in (0, 1)$, $x \in (-1, 1)$ a π-system? ◯ Yes ◯ No Why?

c. (5) Give an example (in formulas or pictures) of an event in $\sigma(R)$ that is *not* measurable over $\sigma(X)$ or $\sigma(Y)$, if possible; if it is not possible, say why.

d. (5) Give an example (in formulas or pictures) of an event in $\sigma(R)$ that is is *not* measurable over $\sigma(X, Y)$, if possible; if it is not possible, say why.

Problem 4. Let $\Omega = (0, 1)$ with Borel sets $\mathcal{F}$, with probability measure given by

$$\mathsf{P}\big((a,\ b]\big) = (1-a)^\beta - (1-b)^\beta, \qquad 0 < a < b < 1$$

for some number $\beta > 0$. Define random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ by

$$X(\omega) := \omega \qquad\qquad S_n(\omega) := \sum_{k=0}^{n} \omega^k$$

a. (4) What is the probability distribution of X? Give either the p.d.f. (carefully) or the name and parameters.

b. (8) For which (if any) $\beta > 0$ does Lebesgue's **monotone convergence theorem** apply to the sequence $\{S_n\}$? What does M.C.T. say?

c. (8) For which (if any) $\beta > 0$ does Lebesgue's **dominated convergence theorem** apply to the sequence $\{S_n\}$? What does D.C.T. say? What is a dominating random variable $Y \in L_1$?

Problem 5. For 2pt each, write your answers in the boxes provided, or circle True or False. No explanations are required. In parts b, c, & d, $g = g(x)$ denotes a completely arbitrary Borel measureable function, not necessarily continuous.

a. If $X \in L_1$ then $X \in L_2$. T F

b. If X is simple[1] and $Y = g(X)$ then Y is simple. T F

c. If X is continuous[2] and $Y = g(X)$ then Y is continuous. T F

d. If X is abs. cont.[3] and $Y = g(X)$ then Y is abs. cont. T F

e. If $\mathsf{E}|X_n| \le 7$ for each $n \in \mathbb{N}$ then $\{X_n\}$ is U.I. T F

f. If $X \in L_4$ and $Y \in L_4$ then for what (if any) p is $X \cdot Y \in L_p$? $\boxed{}$

g. If $X \in L_1$ then $\exp\big(\mathsf{E}[X]\big) \le \mathsf{E}\big[\exp(X)\big]$. T F

h. If $X \in L_1$ and $Y_n := 1_{\{|X|>n\}}$ then $\mathsf{E}[X\,Y_n] \to 0$ as $n \to \infty$. T F

i. If $\mathsf{E}[X^4] = 96$, give an upper bound for $\mathsf{P}[X > 4]$: $\boxed{}$

j. If $X \perp\!\!\!\perp Y$ and $X, Y \in L_2$ then $\mathsf{V}(X - Y) = \mathsf{V}(X) - \mathsf{V}(Y)$ T F

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[1] A *simple* R.V. is one that takes on only finitely-many different values.

[2] This is a sloppy but common way of saying that the random variable has a continuous distribution, *i.e.*, that its C.D.F. $F(x)$ is a continuous function of $x \in \mathbb{R}$.

[3] A distribution is absolutely continuous if it has a density function.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(1-e^{\sigma^2}\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedekor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_1}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta,\gamma)$	$f(x)=\frac{\alpha(x-\gamma)^{\alpha-1}}{\beta^\alpha}e^{-[(x-\gamma)/\beta]^\alpha}$	$x\in(\gamma,\infty)$	$\gamma+\beta\Gamma(1+\alpha^{-1})$		