Final Examination

STA 205: Probability and Measure Theory

Due by Wednesday, 2008 April 30, 2:00 pm

This is an open-book take-home examination.

You must do your own work— no collaboration is permitted. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others (whether or not they are taking this exam). You can reach me by e-mail (`wolpert@stat.duke.edu`) or telephone (919-684-3275).

You must **show** your **work** to get credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible.

This exam is due on or before 2:00 pm Wednesday, 2008 April 30. You may slip it under my office door (211c Old Chem) or hand it to me earlier.

Print Name: ____________________

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Problem 1: Let $\Omega = \mathbb{R}_+ = [0, \infty)$ be the positive half-line, with Borel sets $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$ and probability measure P given by $\mathsf{P}(d\omega) = e^{-\omega}\, d\omega$ or (equivalently)

$$\mathsf{P}\big[(a, b]\big] = e^{-a} - e^{-b} \qquad 0 < a < b < \infty$$

For each integer $n \in \mathbb{N} = \{1, 2, \cdots\}$ define a random variable on $(\Omega, \mathcal{F})$ by

$$X_n(\omega) := \begin{cases} 0 & \text{if } \omega < n \\ 1 & \text{if } \omega \geq n \end{cases}$$

a) Find the mean $m_n = \mathsf{E}[X_n]$ for each $n \in \mathbb{N}$ and the covariance $\Sigma_{mn} = \mathsf{E}[(X_m - m_m)(X_n - m_n)]$ for each $m \leq n \in \mathbb{N}$:

$m_n =$ $\qquad\qquad$ $\Sigma_{mn} =$

b) Give the probability distribution *measure* μ_n of X_n for each n:

Problem 1 (cont'd): As before, $\Omega = \mathbb{R}_+$, $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$, $\mathsf{P}(d\omega) = e^{-\omega}\,d\omega$, and $X_n(\omega) := \mathbf{1}_{[n,\infty)}(\omega)$ for $n \in \mathbb{N}$ (see footnote[1])

c) Give the σ-algebras $\sigma(X_n)$ explicitly for each fixed $n \in \mathbb{N}$:

$$\sigma(X_n) = \left\{ \qquad\qquad\qquad\qquad\qquad\qquad \right\}$$

d) Does the σ-algebra $\mathcal{G} = \sigma(X_1, X_2, ...)$ generated by all the X_n's contain *all* the Borel sets in $\mathbb{R}_+$? If so, say why; if not, find a Borel set $B \in \mathcal{F}$ that is *not* in $\mathcal{G}$.

e) Are X_1 and X_2 independent? Justify your answer.

[1] Recall that the *indicator* random variable $\mathbf{1}_A(\omega)$ is one if $\omega \in A$, otherwise zero.

Problem 2: As in Problem 1, $\Omega = \mathbb{R}_+$, $\mathcal{F} = \mathcal{B}(\mathbb{R}_+)$, $\mathsf{P}(d\omega) = e^{-\omega}\,d\omega$, and $X_n(\omega) := \mathbf{1}_{[n,\infty)}(\omega)$ for $n \in \mathbb{N}$.

a) Prove that the sum $S := \sum_{k=1}^{\infty} X_k$ converges almost surely. Give the name and the mean of the probability distribution of S.

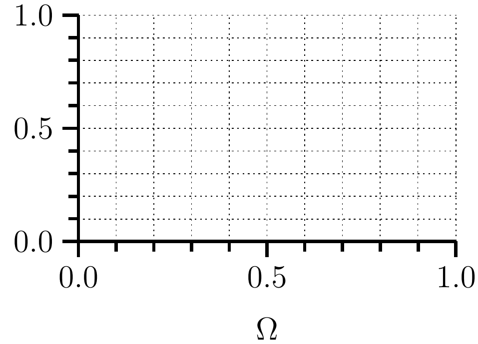
b) Do the partial sums $S_n := X_1 + \cdots + X_n$ converge to S in L_1 as $n \to \infty$? Justify your answer.

c) Set $\mathcal{F}_n = \sigma\{X_1, \cdots, X_n\}$, the σ-algebra generated by the first n of the X_k's. Find the indicated conditional expectations:

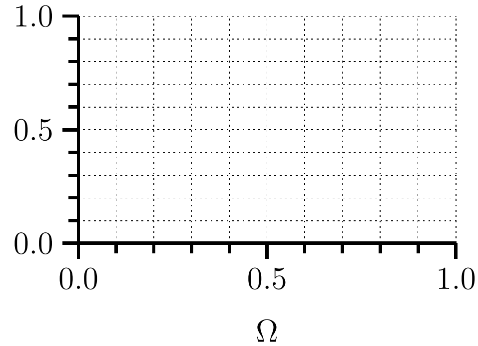
$\mathsf{E}[X_4 \mid \mathcal{F}_2] =$ $\qquad\qquad$ $\mathsf{E}[S \mid \mathcal{F}_2] =$

Problem 3: Let $(\Omega, \mathcal{F}, \mathsf{P}) = \big((0,1], \mathcal{B}, d\omega\big)$ be the unit interval with Lebesgue measure (length). Define three random variables by $X(\omega) := \mathbf{1}_{(0,0.5]}(\omega)$, $Y(\omega) := \mathbf{1}_{(0.3,1]}(\omega)$, and $Z(\omega) := \omega$.

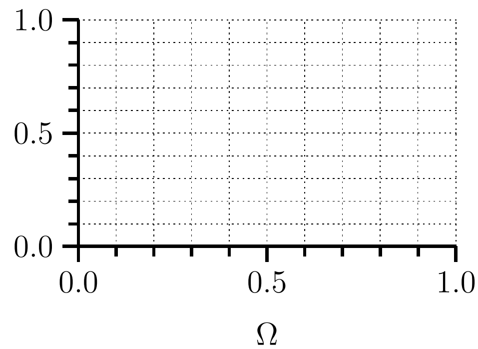
a) Find and plot the conditional expectation $\mathsf{E}[X \mid Y]$:



b) Find and plot the conditional expectation $\mathsf{E}[Z \mid X, Y]$:



c) Find and plot the conditional expectation $\mathsf{E}[W \mid X, Y, Z]$ for the random variable $W(\omega) := \omega^2$:



Problem 4: The random variables $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,1)$ are independent with the Uniform distribution on $(0,1]$.

a) Find the indicated expectations:

$$\mathsf{E}\left[\sin(\pi X_1)\right] = \qquad\qquad \mathsf{E}\left[\frac{1}{\sin(\pi X_1)}\right] =$$

$$\mathsf{E}\left[\frac{1}{\sqrt{X_1}}\right] = \qquad\qquad \mathsf{E}\left[\frac{X_1}{X_2}\right] =$$

b) Set $M_n := \wedge_{k=1}^n X_k = \min\{X_1, \ldots, X_n\}$, the minimum of the first n random variables. Find a sequence $\{a_n\}$ of (nonrandom) real numbers and a non-degenerate probability distribution function G (i.e. one with $0 < G(x) < 1$ for at least one number $x \in \mathbb{R}$) such that

$$\mathsf{P}[a_n M_n \le x] \to G(x)$$

as $n \to \infty$, for every $x \in \mathbb{R}$ where $G(x)$ is continuous.

c) For your sequence a_n above, explain how to evaluate

$$\lim_{n\to\infty} \mathsf{E}[\phi(a_n M_n)]$$

for any continuous bounded function $\phi(x)$.

Problem 5: Short answer questions (no explanations needed) about a probability space $(\Omega, \mathcal{F}, \mathsf{P})$:

a) How many points must Ω have (at minimum) for there to exist n distinct[2] independent events A_j that are "nontrivial" in the sense that $0 < \mathsf{P}[A_j] < 1$ for $j = 1 \ldots n$?

b) How many points must Ω have (at minimum) for there to exist n independent events A_j if we don't insist that they be nontrivial or distinct?

c) How many points must Ω have (at minimum) for there to exist n distinct events A_j if we don't insist that they be independent?

d) If $A_n \supset A_{n+1}$ is a decreasing sequence of events with $\mathsf{P}[\cap_{j=1}^{\infty} A_j] > 0$, does $\mathsf{E}[Y \mid A_n]$ converge, as $n \to \infty$ for every $Y \in L_1(\Omega, \mathcal{F}, \mathsf{P})$?

e) If two events B and C are independent, and if A is independent of both $B \cap C$ and $B^c \cap C$, is it necessarily true that the three events $\{A, B, C\}$ are independent?

[2]Two events A, B are *distinct* if $\emptyset \neq (A \Delta B) := (A \backslash B) \cup (B \backslash A)$.

Problem 6: The *support* of the probability distribution $\mu(dx)$ of an $\mathbb{R}^d$-valued random vector X is the smallest closed set $S \subset \mathbb{R}^d$ for which $\mu(S) = \mathsf{P}[X \in S] = 1$.

a) Let X have a distribution with a continuous density function $f(x)$; show that $\mathsf{E}[\frac{1}{f(X)}]$ is the Lebesgue measure of the support set, $|S| = \int_S dx$ (which may be finite or infinite).

b) If independent random variables $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{Ex}(\lambda)$ have a common exponential distribution with parameter λ (and density function $f(x \mid \lambda) = \lambda e^{-\lambda x} \mathbf{1}_{\mathbb{R}_+}(x)$), find (and justify) the limits:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{f(X_k \mid \lambda)} = \qquad\qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{f(X_k \mid \lambda/2)} =$$

Whoo hoo, you're done!

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	