Midterm Examination

STA 205: Probability and Measure Theory

Wednesday, 2008 Mar 5, 2:50-4:05 pm

This is a closed-book examination. You may use a single sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss questions with others.

Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible. Good luck.

Print Name: ____________________

1.	/20
2.	/20
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4.	/20
5.	/20
6.	/20
Total:	/120

Problem 1. Let X be a random variable with the standard exponential distribution with C.D.F.

$$F(x) := \mathsf{P}[X \le x] = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \ge 0. \end{cases}$$

a. (5) Evaluate the Moment Generating Function $M(t) := \mathsf{E}[e^{tX}]$ for all real numbers $t \in \mathbb{R}$ (be careful!).

   $M(t) =$ ____________________

b. (5) For fixed $0 < t < 1$ and all $a > 0$ use Markov's inequality for e^{tX} to find a bound (which of course will depend on a and t) on

   $\mathsf{P}[X > a] \le$ ____________________

c. (5) Find the *best* bound above, over all possible values of $0 < t < 1$:

   $\mathsf{P}[X > a] \le$ ____________________
   (your bound will depend on a now but no longer on t)

d. (5) How good is this bound? How does it compare to the Chebychev bound for X? To the *actual* value of the function $\mathsf{P}[X > a]$?

Problem 2. For $n \in \mathbb{N}$ let $\{X_i : \; 1 \le i \le n\}$ be random variables on the probability space $(\Omega, \mathcal{F}, \mathsf{P})$, and set $X := \prod_{i=1}^n X_i = X_1 \cdot X_2 \cdot \ldots \cdot X_n$.

a. (5) Give an example of $(\Omega, \mathcal{F}, \mathsf{P})$, n, and $\{X_i\}$ where each $X_i \in L_1(\Omega, \mathcal{F}, \mathsf{P})$ but their product $X \notin L_1(\Omega, \mathcal{F}, \mathsf{P})$:[1]

b. (5) If $\{X_i\}$ are independent, and $t > 0$ is any positive real number, are the random variables $\left\{X_i \mathbf{1}_{\{|X_i| \le t\}}\right\}$ independent too? Why?

[1]Hint: $n = 2$ is enough

Problem 2 (cont).

Still $\{X_i : \ 1 \le i \le n\}$ are random variables and $X := \prod_{i=1}^n X_i$.

c. (5) Prove (for example, by citing an appropriate theorem) that

$$\lim_{t\to\infty} \mathsf{E}\left[\prod_{i=1}^n \Big(|X_i| \wedge t\Big)\right] = \mathsf{E}\left[\prod_{i=1}^n |X_i|\right]$$

d. (5) If each $X_i \in L_1(\Omega, \mathcal{F}, \mathsf{P})$, and if $\{X_i\}$ are independent, complete the proof that their product $X = \prod_{i=1}^n X_i$ is in $L_1(\Omega, \mathcal{F}, \mathsf{P})$, too.

Problem 3. Let P and Q be two probability measues on $(\Omega, \mathcal{F})$ that agree on the collection $\mathcal{C}$ of all **finite** subsets of Ω.

a. (10) If $\Omega = \mathbb{N}$, the positive integers, with $\mathcal{F} = 2^\Omega$ the σ-algebra of all subsets of Ω, does it follow that $\mathsf{P}(A) = \mathsf{Q}(A)$ for every $A \in \mathcal{F}$?
◯ Yes ◯ No Sketch a proof or give a counter-example:

b. (10) If $\Omega = \mathbb{R}_+$, the positive real numbers, with $\mathcal{F} = \mathcal{B}(\Omega)$ the Borel σ-algebra, does it follow that $\mathsf{P}(A) = \mathsf{Q}(A)$ for every $A \in \mathcal{F}$?
◯ Yes ◯ No Sketch a proof or give a counter-example:

Problem 4. Let X_n be independent random variables for $n \in \mathbb{N}$ with

$$\mathsf{P}[X_n = x] = \begin{cases} n^{-1} & x = 1 \\ 1 - n^{-1} & x = 0. \end{cases}$$

a. (4) Does $S_m := \sum_{1 \le n \le m} X_{n^2}$ converge to a finite random variable S as $m \to \infty$? ◯ Yes ◯ No Why or why not?

b. (4) Does $T_m := \sum_{1 \le n \le m} n^{-1} X_n$ converge to a finite random variable T as $m \to \infty$? ◯ Yes ◯ No Why or why not?

c. (4) Does $U_m := \sum_{1 \le n \le m} X_n$ converge to a finite random variable U as $m \to \infty$? ◯ Yes ◯ No Why or why not?

d. (4) Does $V_m := \sum_{1 \le n \le m} 2^n X_{2^n}$ converge to a finite random variable V as $m \to \infty$? ◯ Yes ◯ No Why or why not?

e. (4) Which, if any, of S, T, U, V is in L_1? No proof req'd.

Problem 5. Let $\Omega = \mathbb{R}_+ = (0, \infty)$ with Borel sets $\mathcal{F}$, with probability measure given by

$$\mathsf{P}\big((a,\ b]\big) = e^{-\beta a} - e^{-\beta b}, \qquad 0 \le a \le b < \infty$$

for some number $\beta > 0$. Define random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ by

$$X(\omega) := \omega \qquad Y(\omega) = e^{-\omega} \qquad S_n(\omega) := \sum_{k=0}^{n} \omega^k / k!$$

a. (4) What is the probability distribution of X? Give either the p.d.f. $f(x)$ (correctly for *every* $x \in \mathbb{R}$) or the name and parameter(s).

b. (4) What is the probability distribution of Y? Give either the p.d.f. $f(y)$ (correctly for *every* $y \in \mathbb{R}$) or the name and parameter(s).

c. (6) Find $\mathsf{E}\big[|X|^p\big]$ and $\mathsf{E}\big[|Y|^p\big]$ for all (positive *and* negative) $p \in \mathbb{R}$.

d. (6) For which (if any) $\beta > 0$ does Lebesgue's **dominated convergence theorem** apply to the sequence $\{S_n\}$? What does D.C.T. say? What is a dominating random variable?

Problem 6. For 2pt each, write your answers in the boxes provided, or circle True or False. Each X, X_n, Y is a R.V. on some $(\Omega, \mathcal{F}, \mathsf{P})$; each $\mathcal{F}_i$, $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$; "$\mathsf{V}(\cdot)$" denotes variance. No explanations are required.

a. If $X, Y \in L_2$ then $X^2 + X\,Y \ \in L_1$. T F

b. If $\mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2$, then $(\mathcal{F}_1 \cap \mathcal{G}) \perp\!\!\!\perp (\mathcal{F}_2 \cap \mathcal{G})$. T F

c. If $\mathcal{F}_1 \perp\!\!\!\perp \mathcal{F}_2$, then $(\mathcal{F}_1 \vee \mathcal{G}) \perp\!\!\!\perp (\mathcal{F}_2 \vee \mathcal{G})$. T F

d. If $X \in L_1$ and $X > 0$, then $\mathsf{E}[1/X] < 1/\,\mathsf{E}[X]$. T F

e. If $\{X_n\}$ is U.I then $\{Y_n := {X_n}^2\}$ is U.I. T F

f. If $\{X_n\}$ are independent, then $\sigma\{X_1, X_3, X_5\} \perp\!\!\!\perp \sigma\{X_2, X_4, X_6\}$. T F

g. If $X \perp\!\!\!\perp Y$ then $1/X \perp\!\!\!\perp e^Y$. T F

h. If $X, Y \in L_1$ then $\mathsf{E}[X\, 1_{|Y|>n}] \to 0$ as $n \to \infty$. T F

i. If $\mathsf{E}\big[e^{X^2}\big] \le 10$, give an upper bound for $\mathsf{P}\big[|X| > 3\big]$: [ ]

j. If $X \perp\!\!\!\perp Y$ and $X, Y \in L_2$ then $\mathsf{V}(X - Y) = \mathsf{V}(X) - \mathsf{V}(Y)$ T F

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	