Final Examination

STA 205: Probability and Measure Theory

Due by Wednesday, 2009 April 29, 2:00 pm

This is an open-book take-home examination.

You must do your own work— no collaboration is permitted. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others, whether or not they are taking this exam. You can reach me by e-mail (`wolpert@stat.duke.edu`) or telephone (919-684-3275).

You must **show** your **work** to get full credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible.

This exam is due on or before 2:00 pm Wednesday, 2009 April 29. You may slip it under my office door (211c Old Chem) or hand it to me earlier.

Print Name: ____________________

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Total:	/120

Problem 1: Let $X \in L_2(\Omega, \mathcal{F}, \mathsf{P})$ and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra and $G \subset L_2(\Omega, \mathcal{F}, \mathsf{P})$ the set of all $\mathcal{G}$-measurable square-integrable elements.

a) Prove that there exists a unique (up to a.s. equivalence) element $Y \in G$ that minimizes the distance

$$d(Y, X) = \|X - Y\|_2 = \sqrt{\mathsf{E}|X - Y|^2}$$

b) Prove that the minimizing solution Y above is, in fact, just the conditional expectation, $Y = \mathsf{E}[X \mid \mathcal{G}]$. Thus conditional expectation coincides with orthogonal projection for L_2 random variables.

Problem 2: Let $\Omega = [-1, 1]$, $\mathcal{F} = \mathcal{B}(\Omega)$, $\mathsf{P}(d\omega) = \frac{1}{2}\,d\omega$, and $X(\omega) := \omega^2$.

a) Let $Y(\omega) := e^{\omega}$. Find (and justify) the following:

$\mathsf{E}[X \mid \sigma\{Y\}] =$ $\mathsf{E}[Y \mid \sigma\{X\}] =$

b) Let $Z(\omega) := \mathbf{1}_{\{\omega>0\}}$. Are X and Z independent? ◯ Yes ◯ No
Prove your answer.

c) Let $W(\omega) := \omega$. Find $\mathsf{E}[W \mid \sigma\{X\}] =$
Are X and W independent? ◯ Yes ◯ No Prove your answer.

Problem 3: The random variables $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{Ex}(1)$ are independent with the Exponential distribution on $\mathbb{R}_+$, so $\mathsf{P}[X_n > t] = e^{-t}$ for $t > 0$.

a) If possible, find a Borel function $\phi : \mathbb{R}_+ \to \mathbb{R}$ such that the random variable $Y = \phi(X_1)$ has a non-degenerate distribution (*i.e.*, has a distribution that is not concentrated on just one point) and is independent of X_1; if it is *not* possible, explain why.

b) Set $M_n := \vee_{k=1}^n X_k = \max\{X_1, \ldots, X_n\}$, the maximum of the first n random variables. Find a sequence $\{a_n\}$ of (nonrandom) real numbers and a non-degenerate distribution with DF G such that

$$\mathsf{P}[M_n - a_n \le x] \to G(x)$$

as $n \to \infty$, for every $x \in \mathbb{R}$ where $G(x)$ is continuous.

c) Find (explicitly) the MGF $M(t) = \int e^{tx}\, G(dx)$ or characteristic function $\chi(\omega) = \int e^{i\omega x}\, G(dx)$ for the distribution $G(dx)$ with DF $G(x)$ above.

Problem 4: Let X, Y, and Z be three random variables on the same probability space $(\Omega, \mathcal{F}, \mathsf{P})$, with $X \perp\!\!\!\perp Y$ and $X \perp\!\!\!\perp Z$ ("X is independent of Y and independent of Z").

a) If X, Y, and Z have a Gaussian joint distribution with mean vector $\vec{\mu}$ and invertible covariance matrix Σ, prove that $X \perp\!\!\!\perp \sigma\{Y, Z\}$
(for half-credit, just show that $X \perp\!\!\!\perp (Y - Z)$).

b) What if X, Y, and Z have an arbitrary joint distribution? Either prove that still $X \perp\!\!\!\perp \sigma\{Y, Z\}$, or construct a counter-example where independence fails. Is X independent of $\sigma\{Y, Z\}$? ◯ Yes ◯ No
Is X independent of $(Y - Z)$? ◯ Yes ◯ No

Problem 5: Let $\{X_n\}$ be independent random variables, all with the same distribution with Moment Generating Function $M(t) = \mathsf{E}[e^{tX}]$, and set $S_n := \sum_{k=1}^n X_k$ and $\mathcal{F}_n := \sigma\{X_k : \ k \le n\}$.

a) Prove that $M(2t) \ge M(t)^2$ for every $t \in \mathbb{R}$.

b) Prove that $Y_n := \exp\big\{tS_n - n\log M(t)\big\}$ is an $\mathcal{F}_n$-martingale, for any $t \in \mathbb{R}$ satisfying $M(t) < \infty$.

c) Use the result from b) to evaluate the Moment Generating Function for the first passage time $\tau := \min\{n : \ |S_n| \ge b\}$ for a symmetric simple random walk S_n and $b \in \mathbb{N}$ (Hint: What is $\mathsf{E}Y_{\tau\wedge n}$?)

Problem 6: Short answer questions about a sequence $\{X_n\}$ of random variables on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. Give *brief* explanations, not full proofs. Each question is separate— the conditions don't carry over.

a) If $X_n(\omega) \to X(\omega)$ for some $X \in L_1$ and all $\omega \in \Omega$, and if $X_n \geq 0$, does it follow that $\mathsf{E}X_n \to \mathsf{E}X$?

b) If, for some $X \in L_1$ and all $t > 0$, $\mathsf{P}[|X_n| > t] \leq \mathsf{P}[X > t]$ and $\mathsf{P}[|X_n - X| > t] \to 0$ as $n \to \infty$, does it follow that $\mathsf{E}|X_n - X| \to 0$?

c) If $X_n \searrow 0$ as $n \to \infty$, and $X_1 \in L_1$, does $\mathsf{E}X_n \to 0$?

d) Is it possible to have $\{X_n\}$ non-decreasing *and* independent? ◯ Yes ◯ No

Justify your answer.

e) If $\{X_n\}$ are independent, can $\sup_n X_n$ have a non-degenerate distribution? ◯ Yes ◯ No How about $\limsup_n X_n$? ◯ Yes ◯ No

Name	**Notation**	**pdf/pmf**	**Range**	**Mean μ**	**Variance σ^2**	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	t_ν	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	