Midterm Examination

STA 205: Probability and Measure Theory

Wednesday, 2009 Mar 18, 2:50-4:05 pm

This is a closed-book examination. You may use a single sheet of prepared notes, if you wish, but you may not share materials. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss questions with others.

Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test for this. It is to your advantage to write your solutions as clearly as possible. Good luck.

Print Name: ____________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1. For positive real numbers $\epsilon > 0$, $\alpha > 0$ let X and $\{X_j\}$ be random variables with the Pareto $\mathsf{Pa}(\alpha, \epsilon)$ distribution, with with C.D.F.

$$F(x) := \mathsf{P}[X \le x] = \begin{cases} 0 & -\infty < x \le \epsilon \\ 1 - (x/\epsilon)^{-\alpha} & \epsilon < x < \infty. \end{cases}$$

and hence p.d.f. $f(x) = \alpha\epsilon^\alpha \, x^{-\alpha-1}\mathbf{1}_{\{x>\epsilon\}}$.

a. (5) For which $\alpha, \epsilon > 0$ are $\{X_j\}$ uniformly integrable? Use the definition of U.I. directly.

b. (5) Evaluate the expectation $\mathsf{E}[X^p]$ for all real numbers $p \in \mathbb{R}$. For which p is $X \in L_p$? (Of course this will depend on (α, ϵ))

$\mathsf{E}X^p =$

Problem 1 (cont).

c. (4) For which $\alpha, \epsilon > 0$ does $Y = \sum_{j=1}^{\infty} \mathbf{1}_{\{X_j > j\}}$ converge:

in L_1? ____________________ In L_∞? ____________________
Does it matter whether or not the $\{X_j\}$ are independent? ◯ Yes ◯ No

d. (4) Now assume that $\{X_j\} \overset{\text{iid}}{\sim} \mathsf{Pa}(\alpha, \epsilon)$ are independent, and let $A_n = \{X_1 < ... < X_n\}$ be the event that the first n of them are in increasing order (by convention, $A_0 = A_1 = \Omega$). Is the random variable $Z = \sum_{n=0}^{\infty} \mathbf{1}_{A_n}$ in L_1? ◯ Yes ◯ No If so, find $\mathsf{E}Z$. If not, explain.
(Hint: Finding $\mathsf{P}[A_n]$ is easy, without doing any integration.)

e. (2) Evaluating the expectations of Y and Z in parts c) and d) above relies on two theorems we've learned this year— please name them.

Problem 2. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, and $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra. Let $\mathcal{H}$ be the collection of all events independent[1] of $\mathcal{G}$:

$$\mathcal{H} = \{A \in \mathcal{F} : \ A \perp\!\!\!\perp G \text{ for each } G \in \mathcal{G}\}.$$

a. (10) Is $\mathcal{H}$ a π-system?
◯ Yes ◯ No Sketch a proof or a counter-example with finite Ω:

b. (5) Is $\mathcal{H}$ a λ-system?
◯ Yes ◯ No Sketch a proof or a counter-example with finite Ω:

c. (5) Is $\mathcal{H}$ a σ-algebra? ◯ Yes ◯ No Why?

[1]The notation "$A \perp\!\!\!\perp G$" means $\mathsf{P}[A \cap G] = \mathsf{P}[A]\,\mathsf{P}[G]$.

Problem 3. Let $(\Omega, \mathcal{F}, \mathsf{P})$ be $\Omega = (0, 1]$, the unit interval, with Borel sets $\mathcal{F}$ and Lebesgue measure P, and let X, Y, and $\{X_n\}$ all be L_1 (i.e. **integrable**) random variables on $(\Omega, \mathcal{F}, \mathsf{P})$. Suppose $X_n(\omega) \to X(\omega)$ as $n \to \infty$ for each $\omega \in \Omega$, and let $\{A_n\} \subset \mathcal{F}$ be events.

Indicate which statements below are True or False. If true, indicate which applies: Fatou's Lemma, Lebesgue's dominated (LDC) or monotone (MCT) convergence theorem, the Borel-Cantelli lemma (or Borel's zero-one law), or the inequalities named after Jensen, Minkowski, Hölder, or Markov. If you've forgotten the name of a result or inequality, sketch its statement.

a. (2) T F   If $X_n \geq 0$, then $\liminf_{n\to\infty} \mathsf{E}X_n \geq \mathsf{E}X$.
◯ Fatou ◯ DCT ◯ MCT ◯ Borel ◯ Jensen ◯ Mink ◯ Höld ◯ Markov

b. (2) T F   If $|X_n| \leq 10$, then $\lim_{n\to\infty} \mathsf{E}X_n = \mathsf{E}X$.
◯ Fatou ◯ DCT ◯ MCT ◯ Borel ◯ Jensen ◯ Mink ◯ Höld ◯ Markov

c. (2) T F   If $\mathsf{E}|X_n| \leq 10$, then $\lim_{n\to\infty} \mathsf{E}X_n = \mathsf{E}X$.
◯ Fatou ◯ DCT ◯ MCT ◯ Borel ◯ Jensen ◯ Mink ◯ Höld ◯ Markov

d. (2) T F   If $\|X_n\|_2 \leq 10$, then $\mathsf{P}[|X_n| > t] < 100/t^2$.
◯ Fatou ◯ DCT ◯ MCT ◯ Borel ◯ Jensen ◯ Mink ◯ Höld ◯ Markov

e. (2) T F $\mathsf{E}X \leq \log \mathsf{E}e^X$
◯ Fatou ◯ DCT ◯ MCT ◯ Borel ◯ Jensen ◯ Mink ◯ Höld ◯ Markov

Problem 3 (cont).

Recall that X, Y, and $\{X_n\}$ are L_1 random variables, and $\{A_n\}$ events, with $(\forall\omega\in\Omega)X_n(\omega)\to X(\omega)$. Choose True or False, and indicate reason(s).

f. (2) T F If $\sum_{n<\infty}\mathsf{P}(A_n)<\infty$, then $\mathsf{P}[\cup_{n<\infty}A_n]=0$.
◯ Fatou ◯ DCT ◯ MCT ◯ Borel ◯ Jensen ◯ Mink ◯ Höld ◯ Markov

g. (2) T F If $X_n = X\mathbf{1}_{|X|\le n}$, then $\lim_{n\to\infty}\mathsf{E}X_n=\mathsf{E}X$.
◯ Fatou ◯ DCT ◯ MCT ◯ Borel ◯ Jensen ◯ Mink ◯ Höld ◯ Markov

h. (2) T F If $g(x)$ is bounded & measurable, $\lim_{n\to\infty}\mathsf{E}g(X_n)=\mathsf{E}g(X)$.
◯ Fatou ◯ DCT ◯ MCT ◯ Borel ◯ Jensen ◯ Mink ◯ Höld ◯ Markov

i. (2) T F $\mathsf{E}|X_1X_2|\le\|X_1\|_3\,\|X_2\|_{1.5}$.
◯ Fatou ◯ DCT ◯ MCT ◯ Borel ◯ Jensen ◯ Mink ◯ Höld ◯ Markov

j. (2) T F If $X_n\ge 0$, then $\lim_{n\to\infty}\mathsf{E}X_n=\mathsf{E}X$.
◯ Fatou ◯ DCT ◯ MCT ◯ Borel ◯ Jensen ◯ Mink ◯ Höld ◯ Markov

Problem 4. Let $\Omega = (0, 1]$ with Borel sets $\mathcal{F}$ and probability measure

$$\mathsf{P}(d\omega) := 2\,\omega\, d\omega.$$

Define random variables on $(\Omega, \mathcal{F}, \mathsf{P})$ by

$$X(\omega) := \omega \qquad\qquad S_n(\omega) = \sum_{k=0}^{n-1} (1-\omega)^k, \quad n \in \mathbb{N}$$

a. (4) What is the probability distribution of X? Give its name and parameter(s) (it's on the attached sheet):

b. (8) Give the explicit definition of a random variable $U = U(\omega)$ on $(\Omega, \mathcal{F}, \mathsf{P})$ with the standard uniform distribution $U \sim \mathsf{Un}(0, 1)$:
$U(\omega) =$

c. (8) What does Lebesgue's **dominated convergence theorem** say about the sequence $\{S_n\}$? What $Y \in L_1$ dominates, what is $\|Y\|_1$, and what is the limit S_∞ of $\{S_n\}$?

Problem 5. For 2pt each, write your answers in the boxes provided, or circle True or False. Each X, X_n, Y is a R.V. on some $(\Omega, \mathcal{F}, \mathsf{P})$; each $\mathcal{F}_i$, $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$; "$\mathsf{V}(\cdot)$" denotes variance; $g(x)$ is an arbitrary Borel function. No explanations are required.

a. If $X, Y \in L_2$ then $X^2 Y^2 \in L_1$. T F

b. If $Y := g(X)$ is simple[2] then X is simple too. T F

c. If X has a continuous distribution, then so does $Y := g(X)$. T F

d. If $\mathcal{G} \perp\!\!\!\perp \mathcal{F}_1$ and $\mathcal{G} \perp\!\!\!\perp \mathcal{F}_2$, then $\mathcal{G} \perp\!\!\!\perp (\mathcal{F}_1 \vee \mathcal{F}_2)$. T F

e. If $\mathcal{G} \perp\!\!\!\perp \mathcal{F}_1$ and $\mathcal{G} \perp\!\!\!\perp \mathcal{F}_2$, then $\mathcal{G} \perp\!\!\!\perp (\mathcal{F}_1 \cap \mathcal{F}_2)$. T F

f. If $X \in L_1$, then $\mathsf{E}[\exp(X)] \le \exp(\mathsf{E}[X])$. T F

g. If $\{X_n \ge 0\}$ and $\mathsf{E}X_n \le 100$ then $\left\{Y_n := \sqrt{X_n}\right\}$ is U.I. T F

h. If $X \perp\!\!\!\perp Y$ then $1/X \perp\!\!\!\perp e^Y$. T F

i. If $X \in L_1$ then $\mathsf{E}[X\,\mathbf{1}_{\{Y>n\}}] \to 0$ as $n \to \infty$. T F

j. If $\mathsf{E}\big[2^{X^2}\big] \le 16$, give an upper bound for $\mathsf{P}\big[|X| > 3\big]$: [ ]

[2] A *simple* R.V. is one that takes on only finitely-many different values.

Blank Worksheet

Another Blank Worksheet

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	t_ν	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	