Sta 205 : Homework 1

Due : January 21, 2009

I. **Fields and σ - fields.**

(A) For a three-point outcome set $\Omega = \{a, b, c\}$ and $\mathcal{C} := \Big\{\{a\}\Big\}$, enumerate the class $\aleph$ of all σ-fields $\mathcal{F}$ on Ω that contain $\mathcal{C}$, *i.e.*, satisfy $\mathcal{C} \subset \mathcal{F}$. Also find $\sigma(\mathcal{C})$.

(B) For each integer $n \in \mathbb{N} := \{1, 2, 3, ...\}$, set

$$A_n := \left\{ \frac{m}{n} : \quad m \in \mathbb{N} \right\}.$$

Find $\limsup_{n\to\infty} A_n$ and $\liminf_{n\to\infty} A_n$.

(C) Let f and $\{f_n\}$ be real-valued functions on any set Ω, and let ϵ_k be any decreasing sequence such that $\epsilon_k \searrow 0$ as $k \to \infty$. Show that:

$$\Big\{\omega :\ f_n(\omega) \to f(\omega)\Big\} = \cap_{k=1}^{\infty} \cup_{N=1}^{\infty} \cap_{n=N}^{\infty} \left\{\omega :\ |f_n(\omega) - f(\omega)| < \epsilon_k \right\}.$$

(D) Find a set Ω and two fields $\mathcal{F}_1$ and $\mathcal{F}_2$ for which $\mathcal{F}_1 \cup \mathcal{F}_2$ is *not* a field.

(E) Suppose a collection $\Big\{\mathcal{F}_n :\ n \in \mathbb{N}\Big\}$ of σ-fields satisfies the relation $\mathcal{F}_j \subset \mathcal{F}_{j+1}$ for every $j \in \mathbb{N}$. Does it follow that $\cup \mathcal{F}_j$ is a field? (the answer is "yes"— show why)

(F) Under the same conditions, is $\cup \mathcal{F}_j$ is a σ-field? (this one is "no"— find a counter-example. The idea is to find a sequence $A_n \in \mathcal{F}_n$ with $\cup_n A_n \notin \mathcal{F}_j$ for every j, hence $\cup_n A_n \notin \cup_j \mathcal{F}_j$).

(G) Let $\Omega := (0, 1]$ be the half-open unit interval, and let $n \in \mathbb{N}$ be a FIXED positive integer (say, three). Set

$$\mathcal{B}_n := \Big\{(0, j/2^n],\ j \in \{0, 1, ..., 2^n\}\Big\},$$

the collection of half-open intervals from zero up to an integral multiple of 2^{-n}. Describe in words the σ-field

$$\mathcal{F}_n := \sigma(\mathcal{B}_n)$$

generated by $\mathcal{B}_n$. Tell how many elements $\mathcal{B}_n$ has, and how many elements $\mathcal{F}_n$ has.

II. **Asymptotic Density.**

For any subset $A \subset \mathbb{N}$ of positive integers write $\#(A)$ for its cardinality, *i.e.*, the number of elements it contains. The set A has "asymptotic density" $d(A)$ if the limit

$$d(A) := \lim_{n\to\infty} \frac{\#(A \cap \{1, 2, ..., n\})}{n} \tag{1}$$

exists (*i.e.*, the lim-sup and lim-inf coincide). Denote by

$$\mathcal{A} := \big\{ A \subset \mathbb{N} : \ d(A) \text{ exists } \big\}$$

the collection of all subsets of $\mathbb{N}$ that have an asymptotic density.

(A) Show carefully that the set of even numbers $E := 2\mathbb{N} = \{2, 4, ..\}$ has asymptotic density $d(E) = 1/2$. By "carefully" I mean: compute the ratio in Equation (**??**) *exactly*, for every n (consider even and odd n separately), and compute the limit.

(B) Find a set $B \subset \mathbb{N}$ that does *not* have an asymptotic density.

(C) Find a set $A \in \mathcal{A}$ for which $A \cap E \notin \mathcal{A}$, and conclude that $\mathcal{A}$ is not a field. Hint: the on-line lecture notes might help if you get stuck.