I. \( \sigma \)-algebras and Probability Assignments.

(A) Let \( \{A, B, C\} \subset \mathcal{F} \) be three events in a probability space \((\Omega, \mathcal{F}, \mathbb{P})\). Recall that a partition is a finite or countable collection of disjoint events \( \Lambda_j \in \mathcal{F} \) with \( \bigcup \Lambda_j = \Omega \).

Enumerate all the elements of the partition \( \mathcal{P} = \sigma(A, B, C) \) generated by these events (i.e., \( \mathcal{P} \) is the smallest partition for which \( \{A, B, C\} \subset \sigma(\mathcal{P}) \)). How many (nonempty) elements does \( \mathcal{P} \) have, at most? How many, at minimum?

(B) How many elements does the \( \sigma \)-algebra \( \sigma(\mathcal{P}) \) contain? Describe them in words (don’t list them)

(C) Let’s further assume that the above mentioned events \( A, B, C \) are disjoint with probabilities \( \mathbb{P}(A) = 0.6, \mathbb{P}(B) = 0.3, \mathbb{P}(C) = 0.1 \). Calculate the probability of every event in \( \sigma(A, B, C) \).

II. Fun with null sets.

(A) Let \( \{A_n, n \in \mathbb{N}\} \) be events such that \( \mathbb{P}(A_n) = 0, \forall n \). Show that \( \mathbb{P}(\bigcup_{n=1}^{\infty} A_n) = 0 \).

(B) Let \( \{B_n, n \in \mathbb{N}\} \) be events such that \( \mathbb{P}(B_n) = 1, \forall n \in \mathbb{N} \). What is \( \mathbb{P}(\bigcap_{n=1}^{\infty} B_n) \) ?

(C) Now consider the set of events, \( \{E_\alpha, \alpha \in \mathbb{R}\} \), such that \( \mathbb{P}(E_\alpha) = 0, \forall \alpha \in \mathbb{R} \). Does it necessarily follow that \( \mathbb{P}(\bigcup_{\alpha \in \mathbb{R}} E_\alpha) = 0 \) ? If yes, give a proof, otherwise give a counter example.

(D) Finally, let \( \{B_k\} \) be a collection of events such that, \( \sum_{k=1}^{n} \mathbb{P}(B_k) > n - 1 \). Show that \( \mathbb{P}(\bigcap_{k=1}^{n} B_k) > 0 \) for every \( n \in \mathbb{N} \).

III. Distribution functions and continuity.

(A) Give an example of a function which is continuous on \( \mathbb{R} \), but not uniformly continuous.

(B) Let \( G \) be a continuous distribution function on \( \mathbb{R} \). Show that \( G \) is in fact uniformly continuous. Hint: Consider the points \( \{x_i\} \) for which \( G(x_i) = i/n \) for \( 0 < i < n \).

(C) Now let \( F \) be any distribution function on \( \mathbb{R} \). Show that \( F \) can have at most countably many discontinuities. Hint: Consider the open intervals \((F(x-), F(x))\) for discontinuity points \( x \).
IV. $\pi$ & $\lambda$ - systems.

(A) Let $\Omega = (0, 1] \times (0, 1]$, and consider the following collections of subsets of $\Omega$:

$$\mathcal{A} = \{(0, a] \times (0, b] : 0 < a, b \leq 1\}$$

i. Is $\mathcal{A}$ a $\pi$ - system? Why or why not?
ii. Is $\mathcal{A}$ a $\lambda$ - system? Why or why not?

(B) Consider the following collecton of subsets of the real line:

$$\mathcal{B} = \{(-\infty, b], b \in \mathbb{R}\}$$

i. Show that $\mathcal{B}$ is a $\pi$ - system, but not a $\lambda$ system.
ii. What is the $\lambda$ - system generated by $\mathcal{B}$?

V. $\pi$ - systems and fields.

(A) Let $\mathcal{C}$ be a non empty collection of subsets of $\Omega$, and let $\mathcal{A}(\mathcal{C})$ be the minimal field over $\mathcal{C}$. Show that $\mathcal{A}(\mathcal{C})$ consists of sets of the form

$$\bigcup_{i=1}^{m} \bigcap_{j=1}^{n} A_{ij},$$

where for each pair $(i, j)$ either $A_{ij} \in \mathcal{C}$ or $A_{ij}^{c} \in \mathcal{C}$, and where the $m$ sets $\{B_{i} := \bigcap_{j=1}^{n} A_{ij}, 1 \leq i \leq m\}$, are disjoint. Thus, we can represent explicitly the sets in $\mathcal{A}(\mathcal{C})$, however it turns out that, we cannot do the same for the $\sigma$-field over $\mathcal{C}$.

(B) Now let’s further assume that $\mathcal{C}$ is a $\pi$ system. Show that if $P_1$, $P_2$ are two probability measures which agree on $\mathcal{C}$, then $P_1$, $P_2$ must also agree on $\mathcal{A}(\mathcal{C})$. Hint: Use part(A) and the inclusion-exclusion principle.

(C) Find two probability measures $P_1$, $P_2$ on some set $\Omega$ that agree on a collection of subsets $\mathcal{C}$, but not on $\mathcal{A}(\mathcal{C})$. Obviously (from the previous part) $\mathcal{C}$ cannot be a $\pi$-system. Hint: It’s enough to have $\mathcal{C} = \{A, B\}$ with just two elements, on an outcome space $\Omega$ with just three points.