1. Expectation.

(a) Consider the triangle with vertices \((-1,0), (1,0), (0,1)\) and suppose \((X_1,X_2)\) is a random vector uniformly distributed within this triangle. Compute \(E(X_1 + X_2)\).

(b) Let \((\mathbb{S}, \mathcal{B}, \lambda)\) be a probability space (\(\lambda\) denotes Lebesgue measure and \(\mathcal{B}(\cdot)\) the Borel sets). Let \(X\) be a random variable defined on the probability space described above, with \(X(\omega) := 1_{\mathbb{Q}}\) equal to one on the rationals and zero otherwise. What is \(E(X)\)? Prove it. Note the Riemann integral would be different...

(c) Suppose \(X \in L_1(\Omega, \mathcal{F}, P)\). Show that
\[
\int |X| > n \; X dP \to 0
\]
as \(n\) tends to \(\infty\).

(d) Let \(\{A_n\}\) denote a sequence of events such that \(P(A_n) \to 0\) and let \(X \in L_1\). Show that
\[
\int_{A_n} X dP \to 0
\]

(e) Let \(X \in L_1\), and let \(A\) be an event. Show that
\[
\int_A |X| dP = 0 \text{ if } P(A \cap \{|X| > 0\}) = 0
\]

(f) Let \((\Omega, \mathcal{F}, P)\) be a probability space and \(A_n \in \mathcal{F}, \; n \in \mathbb{N}\). Define a distance measure \(d : \mathcal{F} \times \mathcal{F} \mapsto \mathbb{R}_+\) by \(d(A_n, A_m) \equiv P(A_n \Delta A_m)\),
where “\(\Delta\)” denotes the symmetric difference, \(A \Delta B \equiv (A \setminus B) \cup (B \setminus A)\). Show that, if \(\{A_n\} \subset \mathcal{F}, \; A \in \mathcal{F}\) satisfy \(d(A_n, A) \to 0\), then
\[
\int_{A_n} X dP \to \int_A X dP
\]
for every \(X \in L_1(\Omega, \mathcal{F}, P)\).
2. Convergence Theorems.

(a) Let $X \geq 0$ be a non-negative random variable and define a sequence of real numbers by:
\[ S_n \equiv \sum_{k=0}^{\infty} \frac{k}{2^n} \mathbb{P} \left( \frac{k}{2^n} < X \leq \frac{k+1}{2^n} \right) \quad n \in \mathbb{N}. \]

What is $\lim_{n \to \infty} S_n$? Justify your answer.

(b) Define a sequence of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = ((0, 1], B((0, 1]), \lambda)$ by
\[ X_n \equiv \frac{n}{\log n} 1_{(0, \frac{1}{n})} \quad n \in \mathbb{N}. \]

Show that $X_n \to 0$ almost surely, and $\mathbb{E}(X_n) \to 0$. Also show that the dominated convergence theorem does not apply to this example. Why?

(c) Suppose $\{Y_n\}$ be a sequence of random variables such that
\[ P(Y_n = \pm n^3) = \frac{1}{2n^2}, \quad P(Y_n = 0) = 1 - \frac{1}{n^2}. \]

Show using the Borel-Cantelli lemma that $Y_n \to 0$ almost surely. Compute $\lim_{n \to \infty} \mathbb{E}(Y_n)$. It is 0? Is the Lebesgue Dominated convergence theorem applicable? Why or why not?

(d) Let $\{X_n\}, X$ be random variables, and $0 \leq X_n \to X$. If $\sup_{n} \mathbb{E}(X_n) \leq K < \infty$, then show that $\mathbb{E}(X) \leq K$ and $X \in L_1$.

3. Potpourri.

(a) Let $\{X_n\}$ be a sequence of Bernoulli random variables with
\[ P(X_n = 1) = p_n = 1 - P(X_n = 0) \]

Show that $\sum_{n=1}^{\infty} p_n < \infty$ implies $\sum_{n=1}^{\infty} \mathbb{E}(X_n) < \infty$ and hence conclude that $X_n \to 0$ almost surely.

(b) Let $\{X_n\}$ be a sequence of random variables. Show that
\[ \mathbb{E} \left( \sup_{1 \leq n \leq \infty} |X_n| \right) < \infty \]

if and only if there exists a random variable $0 \leq Y \in L_1$ such that
\[ P(|X_n| \leq Y) = 1, \quad \forall n \geq 1 \]