Sta 205 : Homework #5

Due : February 18, 2009

1. **Indepedence**.

   (a) Let $\{B_i\}$, be independent events. For $N \in \mathbb{N}$ show that

   $$\mathsf{P}\left(\bigcup_{i=1}^{N} B_i\right) = 1 - \prod_{i=1}^{N}[1 - \mathsf{P}(B_i)]$$

   (b) If $\{A_n, n \in \mathbb{N}\}$ is a sequence of events such that

   $$\mathsf{P}(A_n \cap A_m) = \mathsf{P}(A_n)\mathsf{P}(A_m)\ \forall n, m \in \mathbb{N},\ n \neq m,$$

   does it follow that the events $\{A_n\}$ are independent? Give a proof or counter-example.

   (c) Let X be a random variable. Show that X is independent of itself if and only if there is some constant $c \in \mathbb{R}$ for which $\mathsf{P}[X = c] = 1$. Let f be a Borel measurable function, and X a random variable whose distribution is *not* concentrated at a single point. Can $f(X)$ and X be independent? Explain your answer.

   (d) Show that if the event A is independent of the π-system $\mathcal{P}$ and $A \in \sigma(\mathcal{P})$, then $\mathsf{P}(A)$ is either 0 or 1.

   (e) Give a simple example to show that two random variables on the same space $(\Omega, \mathcal{F})$ may be independent according to one probability measure P_1 but dependent with respect to another P_2.

2. **Borel Cantelli**.

   (a) Fix any number $p > 0$ and let $\{X_n\}$ be a sequence of Bernoulli random variables with

   $$\mathsf{P}(X_n = 1) = n^{-p} \qquad \mathsf{P}(X_n = 0) = 1 - n^{-p}.$$

   For $p = 2$ show that the partial sum

   $$S_n := \sum_{k=1}^{n} X_k$$

   converges almost-surely, whether or not the $\{X_n\}$ are independent. If the $\{X_n\}$ are independent, with $p = 1$, does S_n converge? Why or why not?

   (b) Dane draws independent random variables X_n from the uniform distribution on the unit interval $(0, 1]$. Each time he observes a "new record," a random variable X_n larger than any of the previous observations $\{X_k : k < n\}$, he shouts "Whoo hoo!" Show that the probability of a Whoo Hoo shout on the n^{th} draw is $1/n$. Prove that Dane will say Whoo Hoo infinitely-many times. Do you need to assume that the events $A_n :=$ { Whoo Hoo on n^{th} draw } are independent?

(c) Show that the probability of convergence of any sequence of independent random variables is either 0 or 1. Also show that

$$\mathsf{P}[X_n \text{ converges}] = 0$$

if the independent random variables are identically distributed with a non-trivial (i.e., not almost-surely constant) distribution.

(d) Use the Borel-Cantelli lemma to prove that for any sequence of real-valued random variables $\{X_n\}$, there exists constants $c_n \to \infty$ such that

$$\mathsf{P}\left(\lim_{n\to\infty} \frac{X_n}{c_n} = 0\right) = 1.$$

Find the numbers c_n explicitly in terms of the CDF functions

$$F_n(x) = \mathsf{P}[X_n \le x], \quad x \in \mathbb{R}.$$

Find a suitable sequence $\{c_n\}$ explicitly for an i.i.d. sequence $\{X_n\} \overset{\text{iid}}{\sim} \mathsf{No}(0,1)$ of standard Gaussian random variables to ensure that $X_n/c_n \to 0$ almost surely.

3. **Mixed Bag**.

(a) Suppose $\{A_n, n \in \mathbb{N}\}$ are independent events satisfying $\mathsf{P}(A_n) < 1$, $\forall n \in \mathbb{N}$. Show that $\mathsf{P}(\bigcup_{n=1}^{\infty} A_n) = 1$ if and only if $\mathsf{P}(A_n \text{ i.o.}) = 1$. Give an example to show that the condition $\mathsf{P}(A_n) < 1$ cannot be dropped.

(b) Suppose $\{A_n\}$ is a sequence of events. If $\mathsf{P}(A_n) \to 1$ as $n \to \infty$, prove that there exists a subsequence $\{n_k\}$ tending to infinity such that $\mathsf{P}(\cap_k A_{n_k}) > 0$.

(c) Let A_n be a sequence of events. If there exists $\epsilon > 0$ such that $\mathsf{P}(A_n) \ge \epsilon$ for all $n \in \mathbb{N}$, does it follow that there exists a subsequence $\{n_k\}$ tending to infinity such that $\mathsf{P}(\cap_k A_{n_k}) > 0$? Why or why not?

(d) Let $\{X_n\}$ be i.i.d. random variables, with tail σ-field

$$\mathcal{T} \equiv \bigcap_n \mathcal{F}_{\ge n}, \qquad \mathcal{F}_{\ge n} \equiv \sigma\{X_m : m \ge n\}$$

Fix some $B > 0$. Is the event

$$\begin{aligned} E_B &= \{|X_n| \le B \text{ for infinitely-many } n\} \\ &= \cap_{m=1}^{\infty} \cup_{n=m}^{\infty} \{\omega : |X_n(\omega)| \le B\} \end{aligned}$$

in $\mathcal{T}$? Prove or disprove it. Express its probability, $\mathsf{P}[E_B]$, in terms of the random variables' common CDF, $F(x) \equiv \mathsf{P}[X_n \le x]$. Be careful.