1. Independence.

(a) Let \( \{B_i\} \) be independent events. For \( N \in \mathbb{N} \) show that
\[
P \left( \bigcup_{i=1}^{N} B_i \right) = 1 - \prod_{i=1}^{N} [1 - P(B_i)]
\]

(b) If \( \{A_n, n \in \mathbb{N}\} \) is a sequence of events such that
\[
P(A_n \cap A_m) = P(A_n)P(A_m) \quad \forall n, m \in \mathbb{N}, \, n \neq m,
\]
does it follow that the events \( \{A_n\} \) are independent? Give a proof or counter-example.

(c) Let \( X \) be a random variable. Show that \( X \) is independent of itself if and only if there is some constant \( c \in \mathbb{R} \) for which \( P[X = c] = 1 \). Let \( f \) be a Borel measurable function, and \( X \) a random variable whose distribution is not concentrated at a single point. Can \( f(X) \) and \( X \) be independent? Explain your answer.

(d) Show that if the event \( A \) is independent of the \( \pi \)-system \( \mathcal{P} \) and \( A \in \sigma(\mathcal{P}) \), then \( P(A) \) is either 0 or 1.

(e) Give a simple example to show that two random variables on the same space \((\Omega, \mathcal{F})\) may be independent according to one probability measure \( P_1 \) but dependent with respect to another \( P_2 \).

2. Borel Cantelli.

(a) Fix any number \( p > 0 \) and let \( \{X_n\} \) be a sequence of Bernoulli random variables with
\[
P(X_n = 1) = n^{-p} \quad P(X_n = 0) = 1 - n^{-p}.
\]
For \( p = 2 \) show that the partial sum
\[
S_n := \sum_{k=1}^{n} X_k
\]
converges almost-surely, whether or not the \( \{X_n\} \) are independent. If the \( \{X_n\} \) are independent, with \( p = 1 \), does \( S_n \) converge? Why or why not?

(b) Dane draws independent random variables \( X_n \) from the uniform distribution on the unit interval \((0, 1]\). Each time he observes a “new record,” a random variable \( X_n \) larger than any of the previous observations \( \{X_k : k < n\} \), he shouts “Whoo hoo!” Show that the probability of a Whoo Hoo shout on the \( n^{th} \) draw is \( 1/n \). Prove that Dane will say Whoo Hoo infinitely-many times. Do you need to assume that the events \( A_n := \{ \text{Whoo Hoo on } n^{th} \text{ draw} \} \) are independent?
(c) Show that the probability of convergence of any sequence of independent random variables is either 0 or 1. Also show that

\[ P[X_n \text{ converges}] = 0 \]

if the independent random variables are identically distributed with a non-trivial (i.e., not almost-surely constant) distribution.

(d) Use the Borel-Cantelli lemma to prove that for any sequence of real-valued random variables \( \{X_n\} \), there exists constants \( c_n \to \infty \) such that

\[ P \left( \lim_{n \to \infty} \frac{X_n}{c_n} = 0 \right) = 1. \]

Find the numbers \( c_n \) explicitly in terms of the CDF functions

\[ F_n(x) = P[X_n \leq x], \quad x \in \mathbb{R}. \]

Find a suitable sequence \( \{c_n\} \) explicitly for an i.i.d. sequence \( \{X_n\} \sim \text{No}(0,1) \) of standard Gaussian random variables to ensure that \( X_n/c_n \to 0 \) almost surely.

3. **Mixed Bag.**

(a) Suppose \( \{A_n, n \in \mathbb{N}\} \) are independent events satisfying \( P(A_n) < 1, \forall n \in \mathbb{N} \). Show that \( P(\bigcup_{n=1}^{\infty} A_n) = 1 \) if and only if \( P(A_n \text{ i.o.}) = 1 \). Give an example to show that the condition \( P(A_n) < 1 \) cannot be dropped.

(b) Suppose \( \{A_n\} \) is a sequence of events. If \( P(A_n) \to 1 \) as \( n \to \infty \), prove that there exists a subsequence \( \{n_k\} \) tending to infinity such that \( P(\cap_k A_{n_k}) > 0 \).

(c) Let \( A_n \) be a sequence of events. If there exists \( \epsilon > 0 \) such that \( P(A_n) \geq \epsilon \) for all \( n \in \mathbb{N} \), does it follow that there exists a subsequence \( \{n_k\} \) tending to infinity such that \( P(\cap_k A_{n_k}) > 0 \)? Why or why not?

(d) Let \( \{X_n\} \) be i.i.d. random variables, with tail \( \sigma \)-field

\[ T = \bigcap_n F_{\geq n}, \quad F_{\geq n} = \sigma\{X_m : m \geq n\} \]

Fix some \( B > 0 \). Is the event

\[ E_B = \{|X_n| \leq B \text{ for infinitely-many } n\} \]

= \( \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} \{\omega : |X_n(\omega)| \leq B\} \)

in \( T \)? Prove or disprove it. Express its probability, \( P[E_B] \), in terms of the random variables’ common CDF, \( F(x) \equiv P[X_n \leq x] \). Be careful.