Sta 205 : Home Work #6

Due : 2009 Mar 04

1. **Fubini and Tonelli**.

   (a) Let X be a positive random variable (i.e, $X \geq 0$ a.s) on $(\Omega, \mathcal{F}, \mathsf{P})$. Show that

   $$\mathsf{E}(X) = \int_0^\infty \mathsf{P}(X > t)\ dt$$

   (note X need not have an absolutely-continuous distribution). Also verify that for any $\alpha > 0$

   $$\mathsf{E}(X^\alpha) = \int_0^\infty \alpha t^{\alpha-1} \mathsf{P}(X > t) dt$$

   (b) Define probability spaces $(\Omega_i, \mathcal{B}_i, \mu_i)$, for $i = 1, 2$ as follows. Let each $\Omega_i := (0, 1]$ be the unit interval, with σ-algebras

   $$\mathcal{B}_1 = \text{ Borel sets of } (0,1] \qquad \mathcal{B}_2 = \text{ All subsets of } (0,1],$$

   and let μ_1 be Lebesgue measure and μ_2 counting measure– so that $\mu_1(A)$ is the length of any set $A \in \mathcal{B}_1$ and $\mu_2(A)$ is the number of points in $A \in \mathcal{B}_2$. Define

   $$f(x, y) := \mathbf{1}_{x=y}(x, y)$$

   Set

   $$I_1 := \int_{\Omega_1} \left[\int_{\Omega_2} f(x, y) \mu_2(dy) \right] \mu_1(dx) \qquad I_2 := \int_{\Omega_2} \left[\int_{\Omega_1} f(x, y) \mu_1(dx) \right] \mu_2(dy)$$

   Compute I_1 and I_2. Is $I_1 = I_2$? Are the measures μ_1 and μ_2 σ-finite? Why doesn't Fubini's theorem hold here?

   (c) This problem is a probabilistic version of the familiar integration-by-parts formula from calculus. Suppose F and G are two distribution functions with no common points of discontinuity in an interval $(a, b]$. Show that

   $$\int_{(a,b]} G(x) F(dx) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} F(x) G(dx)$$

   Show that the above formula need not hold true if F and G have common discontinuities.

2. **Uniform Integrability (UI)**

A family $\{X_\alpha\}$ of random variables is called *uniformly integrable* (or UI) if

$$\mathsf{E}\Big[|X_\alpha|\,\mathbf{1}_{\{|X_\alpha|>t\}}\Big] \to 0$$

as $t \to \infty$, uniformly in α.

(a) Let $\{X_n\}$ be a sequence of iid, L_1 random variables. Set $S_n \equiv \sum_{i=1}^n X_i$. Show that the sequence of random variables $\{Y_n\}$ defined by $Y_n \equiv S_n/n$ is UI.

(b) Let $X_n \sim \mathsf{No}(0, \sigma_n^2)$. Find a simple condition on $\{\sigma_n^2\}$ such that $\{X_n\}$ is UI.

(c) If $\{X_n\}$ and $\{Y_n\}$ are UI, show that so is $\{X_n + Y_n\}$.

(d) Suppose $\{X_n, n \geq 1\}$ is an **arbitrary** sequence of non-negative random variables, and set $M_n \equiv \vee_{i=1}^n X_i$. If $\{X_n\}$ is UI, show that $\mathsf{E}(M_n)/n \to 0$.

(e) Let $\phi(x)$ be a function which grows faster than x at infinity, i.e, $\phi(x)/x \nearrow \infty$ (monotonically) as $x \to \infty$. Let $\mathcal{C}$ be a collection of random variables such that, for some fixed $B < \infty$ and all $Z \in \mathcal{C}$,

$$\mathsf{E}\Big(\phi(|Z|)\Big) \leq B.$$

Show that $\mathcal{C}$ is UI. What is the implication for $\phi(x) = x^2$?

3. **Convergence Theorems Revisited**.

(a) Let X be a non-negative real valued random variable. Show that:

i. $\lim_{n\to\infty} n\mathsf{E}(\frac{1}{X}\mathbf{1}_{[X>n]}) = 0$.

ii. $\lim_{n\to\infty} n^{-1}\mathsf{E}(\frac{1}{X}\mathbf{1}_{[X>n^{-1}]}) = 0$.

(b) Suppose $\{p_k, k \geq 0\}$ is a probability mass function on $\{0, 1, ...\}$ and define the generating function

$$P(z) = \sum_{k=0}^{\infty} p_k z^k \qquad 0 \leq z \leq 1$$

Prove using Dominated Convergence theorem that

$$\frac{d}{dz}P(z) = \sum_{k=1}^{\infty} p_k k z^{k-1} \qquad 0 \leq z \leq 1.$$

Note you may wish to consider the cases $z < 1$ and $z = 1$ separately. What is $P'(1)$? $P'(0)$? Can you express the variance of a random variable X, if it exists, in terms of $P(z)$ and its derivatives? How about $p_k = \mathsf{P}[X = k]$?