Sta 205 : Home Work #7
Due : March 16, 2009

I. True or False? Answer whether each of the following statements is true or false. If your answer is true, answer why it is true. If it is false, show why— perhaps by giving a simple counter example.

(A) If \{X_n, \ n \in \mathbb{N}\} is a uniformly integrable (U.I.) collection of random variables, then \(X_n \in L_1\) for each \(n\).

(B) Define a sequence \(\{X_n\}\) of random variables on the unit interval with Lebesgue measure, \((\Omega, \mathcal{F}, \mathbb{P})\) with \(\Omega = (0,1]\), \(\mathcal{F} = \mathcal{B}\), and \(\mathbb{P} = \lambda\), by \(X_n \equiv \sqrt{n}1_{(0,\frac{1}{n}]}.\) Then \(\{X_n\}\) is UI.

(C) Let \(\{X_n\}\) be a sequence of random variables for which \(e^{\left|X_n\right|}\) is uniformly bounded in \(L_1\), i.e., satisfies \(Ee^{\left|X_n\right|} \leq B\) for some \(B < \infty\) and all \(n\). Then \(\{X_n\}\) is UI.

(D) Let \(\{X_n\}\) be a sequence of random variables that is uniformly bounded in \(L_1\), i.e., satisfies \(E\left|X_n\right| \leq B\) for some \(B < \infty\) and all \(n\). Then \(\{X_n\}\) is UI.

II. Characteristic Functions.

(A) Let \(X\) be a random variable, and define
\[ \phi_X(\omega) \equiv E(e^{i\omega X}), \quad \omega \in \mathbb{R} \]
Show that \(\phi_X(\omega)\) is uniformly continuous in \(\mathbb{R}\).

(B) Find the characteristic functions of the following random variables:

i. \(X \sim \text{Ge}(p)\)

ii. \(Y \sim \text{Ex}(\lambda)\)

iii. \(Z = X/n, \quad X \sim \text{Ge}(\lambda/n)\)

Find the limit of \(\phi_Z(\omega)\) from part (iii) above as \(n \to \infty\). Recognize it?

III. Infinite Divisibility.
The distribution of a random variable \(X\) is called infinitely divisible if, for every \(n \in \mathbb{N}\), there exist \(n\) i.i.d random variables \(\{Y_i\}\) such that \(X\) has the same distribution as \(\sum_{i=1}^{n} Y_i\). Use characteristic functions to show that if \(X \sim \text{Po}(\lambda)\), then \(X\) is infinite divisible. (Hint: Recall that if random variables \(\{Y_i\}\) are independent then \(\phi_{\sum Y_i}(\omega) = \prod \phi_{Y_i}(\omega)\) for all \(\omega \in \mathbb{R}\))

---

1Starting at \(x = 0\)— with p.m.f. \(f(x \mid p) = pq^x, \ x = 0,1,2,...\)
2Rate parametrization— with p.d.f. \(f(y) = 1\ee^{-y}, y > 0.\)