Sta 205 : Home Work #9

Due : April 1, 2009

I. **Convergence Concepts: a.s. and i.p.**

(A) Let $\{X_n\}$ be a monotonically increasing sequence of RVs such that $X_n \to X$ in probability (i.p). Show that $X_n \to X$ almost surely (a.s).

(B) Let $\{X_n\}$ be any sequence of RVs. Show that $X_n \to X$ a.s. if and only if

$$\sup_{k \ge n} |X_k - X| \to 0 \qquad \text{i.p.}$$

(C) Let $\{X_n\}$ be an arbitrary sequence of RVs and set $S_n = \sum_{i=1}^n X_i$. Show that $X_n \to 0$ a.s. implies that $S_n/n \to 0$ a.s.

(D) Let $\{X_n\}$, $\{Y_n\}$ be two sequences of RVs such that $0 \le X_n \le Y_n$ and $Y_n \to 0$ i.p. Show that $X_n \to 0$ i.p.

(E) Suppose $\{X_n\}$ are identically distributed with finite variance. Fix $\epsilon > 0$. Show that $n\,\mathsf{P}\left[|X_1| \ge \epsilon\sqrt{n}\right] \to 0$. Also show that $\frac{\bigvee_{i=1}^n |X_i|}{\sqrt{n}} \to 0$ i.p., where "$\bigvee a_i$" denotes the maximum of $\{a_i\}$.

(F) For random variables X, Y define

$$\rho(X,Y) \equiv \mathsf{E}\frac{|X-Y|}{1+|X-Y|}.$$

The function ρ is a metric (you do not have to prove that), i.e., it's non-negative, symmetric, satisfies the triangle inequality, and vanishes if and only if $X = Y$. Show that ρ "metrizes" convergence in probability: i.e., $X_n \to X$ i.p., if and only if $\rho(X_n, X) \to 0$.

II. L_p **Convergence**

(A) Let $\{X_n\}$ be a sequence of positive RVs such that $X_n \to X$ i.p. and $\mathsf{E}(X_n) \to \mathsf{E}(X)$. Show that $X_n \to X$ in L_1.

(B) For any two events A and B, define the distance $d(A,B)$ as

$$d(A,B) \equiv \mathsf{P}(A \Delta B)$$

where (as usual) $A\Delta B \equiv (A \cap B^c) \cup (A^c \cap B)$ denotes the symmetric difference. Prove that for any sequence of events $\{A_n\}$, $d(A_n, A) \to 0$ if and only if the indicator functions satisfy $\mathbf{1}_{A_n} \to \mathbf{1}_A$ in L_2.

(C) Give an example of a sequence of RVs $\{X_n\} \subset L_2$ which converge in L_1 but do not converge in L_2.

(D) Let $(\Omega, \mathcal{F}, \mathsf{P}) = ((0,1], \mathcal{B}(0,1], \lambda)$ be the unit interval with Lebesgue measure, and define $X_n(\omega) \equiv \omega^n$, $\omega \in \Omega$. For what $p \in [1, \infty]$, does the sequence $\{X_n\}$ converge in L_p ? If it does converge for some $p \in [1, \infty]$, find the limiting random variable (it might depend on p). Explain your answer.

III. **More on L_p**

(A) For a random variable X, $1 < p < q < \infty$, show that

$$0 \leq ||X||_1 \leq ||X||_p \leq ||X||_q \leq ||X||_\infty$$

(B) For $1 < p < q < \infty$, show that

$$L_\infty \subset L_q \subset L_p \subset L_1$$

where L_p denotes the space of all RVs X with $||X||_q < \infty$. Hint: Jensen's inequality might be needed here.

(C) Show the following form of Hölder's inequality : For RVs X, Y

$$\mathsf{E}(|XY|) \leq ||X||_1 ||Y||_\infty$$

(D) Show the following form of Minkowski's inequality: For RVs X, Y

$$||X + Y||_\infty \leq ||X||_\infty + ||Y||_\infty$$

(E) If $X \in L_3(\Omega, \mathcal{F}, \mathsf{P})$ and $Y \in L_6(\Omega, \mathcal{F}, \mathsf{P})$, for what $r \in (0, \infty)$ is $X \cdot Y \in L_r$? Why? Give a bound for $\|X \cdot Y\|_r$.