Sta 205 : Homework #11

Due : April 15, 2009

I. **Convergence In Distribution**

(A) For events $\{A_n\}$ and A in some probability space $(\Omega, \mathcal{F}, \mathsf{P})$, define Bernoulli random variables by $X_n \equiv 1_{A_n}$ and $X \equiv 1_A$. As $n \to \infty$,

i. Under what conditions on $\{A_n\}$ and A will $X_n \Rightarrow X$?

ii. Under what conditions on $\{A_n\}$ and A will $X_n \to X$ in L_1?

iii. Under what conditions on $\{A_n\}$ and A will $X_n \to X$ in L_∞?

(B) Let $\{X_n\}$ be a sequence of RV's with distributions given by

$$\mathsf{P}\left[X_n = 1 - \frac{1}{n}\right] = \mathsf{P}\left[X_n = 1 + \frac{1}{n}\right] = \frac{1}{2}.$$

Show that X_n converges in distribution, and find the limiting distribution.

(C) Define probability density functions by

$$f_n(x) = \begin{cases} 1 - \cos(2n\pi x) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

and let $\mu_n(dx) = f_n(x)\,dx$ be the corresponding distributions— so $\mu_n(A) = \mathsf{P}[X_n \in A]$, if X_n has p.d.f. $f_n(x)$. Show that μ_n converges weakly and find the weak limit. Also show that the density functions f_n do *not* converge pointwise.

(D) Let $Y_n \sim \mathsf{No}(\mu_n, \sigma_n^2)$ and $Y \sim \mathsf{No}(\mu, \sigma^2)$ be normally-distributed random variables. Show that $Y_n \Rightarrow Y$ if and only if $\mu_n \to \mu$ and $\sigma_n^2 \to \sigma^2$.

II. **Central Limit Theorem** (CLT)

(A) Fix $a > 1$ and let X_n be an i.i.d. sequence with density function

$$f(x) = a|x|^{-1-2a}, \quad |x| \ge 1; \qquad f(x) = 0, \quad |x| < 1.$$

Compute $\mathsf{E}[X_1]$ and $\mathsf{E}[X_1^2]$. Set $S_n \equiv \sum_{i=1}^n X_i$. For what number(s) $p \in \mathbb{R}$ does S_n/n^p have a non-trivial limiting distribution as $n \to \infty$? What is that distribution? Extra credit: What happens for $0 < a < 1$? For $a = 1$?

(B) **Delta method.** Let $\{X_j\} \overset{\text{iid}}{\sim} \mathsf{Bi}(1, \theta)$ be independent Bernoulli random variables with partial sum $S_n \equiv \sum_{j \le n} X_j \sim \mathsf{Bi}(n, \theta)$ and sample mean $\overline{X}_n \equiv S_n/n$, for some $\theta \in (0, 1)$, and let $\phi \in \mathcal{C}^3(0, 1)$ be a real-valued function on the unit interval with three continuous derivatives at every point. For large n use Taylor's theorem to find the approximate mean and variance of $\phi(\overline{X}_n)$, **correct to order** $1/n$. Show your work; keep track of the remainder terms!