Sta 114 = Mth 136 : Homework 3

1. SD memory cards are used to store data in phones, cameras, and computers. The cards have error-correcting circuitry so they will appear to work perfectly when in fact they have up to eight defective memory cells. The numbers of defects in SD cards have Poisson distributions with a mean θ that varies from one manufacturing lot to another. Five cards are taken at random from one particular lot and their defect counts are

$$\mathbf{x} = \{2,\ 3,\ 5,\ 0,\ 0\}$$

   If θ has a Gamma prior distribution with mean $\mu = 0.10$ and standard deviation $\sigma = 0.10$, find the posterior distribution of θ for this lot.

2. Let $\{X_1, \ldots, X_n\}$ be n iid random variables, each with pdf

$$f(x \mid \theta) = \begin{cases} \theta x^{\theta-1} & 0 < x < 1 \\ 0 & \text{other } x. \end{cases}$$

   and suppose that the unknown value of the parameter $\theta > 0$ has a Gamma $\mathsf{Ga}(\alpha, \beta)$ prior distribution for some fixed numbers $\alpha > 0$, $\beta > 0$. Find the posterior mean and variance of θ.

3. In a sequential clinical trial, let the probability of successful outcome θ have a uniform prior distribution on $(0, 1)$. Suppose that the first patient has a successful outcome. Find the Bayes estimate of θ that would be be obtained for each of

   (a) Squared-error loss, $L(\theta, a) = |\theta - a|^2$;

   (b) Absolute error loss, $L(\theta, a) = |\theta - a|$.

4. If squared-error loss is used, what is the Bayes estimate of the proportion θ of defective items in a large shipment, if β has a beta $\mathsf{Be}(\alpha = 2, \beta = 8)$ prior distribution and if a random sample of $n = 20$ items from the shipment is found to contain precisely two defective items?

5. Using squared-error loss, find the Bayes estimate of the mean number θ of defects in SD memory cards, after observing defect counts of

$$\mathbf{x} = \{2,\ 3,\ 5,\ 0,\ 0\}$$

in a random sample of size five, if (again) $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Po}(\theta)$ and θ has a gamma prior distribution with mean $\mu = 0.10$ and standard deviation $\sigma = 0.10$ (recall exercise (1) above).

6. If a rare event (like a volcanic eruption or flood) occurs with small probability p each year, then the number of years (rounded up) until such an event occurs has a discrete distribution with pf

$$\Pr[X_j = x] = \begin{cases} pq^{x-1} & x = 1, 2, \ldots \\ 0 & \text{other } x \end{cases}$$

whose expectation $\mathsf{E}[X_j]$ is called the "return rate" $\theta = 1/p$. Note that each $X_i \geq 1$, since we are counting the total number of years (or parts of years) needed, *including* that in which the rare event occurs. Upon observing a random sample of waiting times (in years)

$$\mathbf{x} = \{\ 10,\ 20,\ 40,\ 50\ \},$$

find the Bayes estimate $\delta(\mathbf{x})$ of the return rate θ, under squared error loss using a uniform prior distribution for p on the unit interval $(0, 1)$ (what prior does that induce for θ?).

7. Suppose that $\mathbf{x} = \{X_1, \ldots, X_n\}$ is a random sample of size n from the exponential distribution with rate parameter $\beta > 0$. Find the MLE $\hat{\beta}$.

8. Let $\mathbf{x} = \{X_1, \ldots, X_n\}$ be a random sample of size n from a continuous distribution with pdf

$$f(x \mid \theta) = e^{\theta - x} \mathbf{1}_{x > \theta}$$

and suppose that $\theta \in \Theta = (-\infty, \infty)$ is unknown.

   (a) Show that the MLE $\hat{\theta}(\mathbf{x})$ does not exist.

   (b) Find another *version* of the pdf (*i.e.*, another function with the same integrals over any interval) for which the MLE *does* exist, and find $\hat{\theta}(\mathbf{x})$.