Sta 114 = Mth 136 : Homework 6

Introduction

This problem set concerns the normal distribution and some close relatives, the χ^2_ν (or, equivalently, $\mathsf{Ga}(\nu/2, 1/2)$) and t_ν. Problem (1) is conceptually routine but some will find tough sledding in the calculus; it's here because changes of variables like this one (and like $\theta = \log p/(1-p)$ or $p = b/(1+b)$) come up a lot and some practice here will help you later. Problem (8) is a specific version of an important general problem— finding the posterior distribution for (at least approximately) normal data of unknown mean and variance, with a conjugate prior distribution. Those two (first and last) seem harder than the others, to me. Good luck!

Problems

1. The standard "Student t distribution with ν degrees of freedom" has a probability density function of the form

$$f_\nu(x) = \frac{c}{(1 + x^2/\nu)^{(\nu+1)/2}}, \qquad -\infty < x < \infty$$

for some $c > 0$ (the word "standard" here means that it is centered at zero and has unit scale; if X has the standard t distribution, then $Y = m + sX$ has the t distribution with scale s, centered at m). Find c and find the variance of the t distribution by evaluating the integral

$$I(p) \equiv \int_{-\infty}^{\infty} |x|^p \, (\nu + x^2)^{-(\nu+1)/2} \, dx = 2 \int_0^\infty x^p \, (\nu + x^2)^{-(\nu+1)/2} \, dx \quad (1)$$

for $p = 0$ and $p = 2$. A good way to do that is to change variables from x to $z = (1 + x^2/\nu)^{-1} = \nu/(\nu + x^2)$ (what is the range of z as x goes from 0 to ∞? What is $(1-z)$? How about $z^{-1}(1-z)$? Can you write x as a product of powers of z and $(1-z)$ and a constant? What is dx in terms of z and dz? What is x^p in terms of z?), then use the fact that

$$\int_0^1 z^{\alpha-1}(1-z)^{\beta-1} \, dz = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

for any numbers $\alpha > 0$, $\beta > 0$, which you know from the form of the normalizing constant for the beta distribution. Note that the variance might be infinite if ν is too small for the integral (1) to be finite. You should find that the variance is *always* bigger than one, the variance of the standard normal distribution.

2. Find the limit as $\nu \to \infty$ of the un-normalized t density,

$$\frac{1}{(1 + x^2/\nu)^{(\nu+1)/2}}$$

Do you recognize it?

3. The t distribution was introduced (both by our text book authors and, early last century, by William Gossett who invented it and published about it under the pseudonym *Student*) as a ratio

$$t = \frac{Z}{\sqrt{Y/\nu}}$$

for independent random variables $Z \sim \mathsf{No}(0, 1)$ and $Y \sim \chi^2_\nu$. Show that t^2 is the ratio of two independent Gamma-distributed random variables, each with mean one. Find the parameters for each Gamma distribution.

4. Let $\{X_1, \ldots, X_n\} \overset{\text{iid}}{\sim} \mathsf{No}(\mu, \sigma^2)$ be a simple random sample from the Normal distribution with unknown mean μ but **known** variance σ^2. Denote by $\Phi(z)$. For any $0 < \gamma < 1$ let z^*_γ be the $(1 + q)/2$'th quantile of the $\mathsf{No}(0, 1)$ distribution, so $\Phi(z^*_\gamma) = \frac{1+\gamma}{2}$ and $\Phi(-z^*_\gamma) = \frac{1-\gamma}{2}$. If $\bar{x}_n$ denotes the sample mean of $\{X_i\}$, show that

$$\left(\bar{x}_n - \frac{z^*_\gamma\, \sigma}{\sqrt{n}}, \quad \bar{x}_n + \frac{z^*_\gamma\, \sigma}{\sqrt{n}}\right)$$

is a confidence interval for μ with coefficient γ.

5. The eight observations $\{31, 35, 26, 34, 38, 30, 29, 25\}$ are taken from a normal $\mathsf{No}(\mu, \sigma^2)$ distribution with both μ and σ unknown. Find the shortest confidence interval for μ with coefficient $\gamma = 0.80$ and (separately) with $\gamma = 0.98$.

6. For n observations $\{X_1, \ldots, X_n\}$ from a normal $\mathsf{No}(\mu, \sigma^2)$ distribution with both μ and σ unknown, let L_γ be the shortest confidence interval for μ that can be constructed with coefficient γ. For each n, γ, μ, and σ find the expected *squared* length $\mathsf{E}L^2_\gamma$ of the shortest confidence interval of index γ. Denote by t^γ_ν the $(1+\gamma)/2$th quantile of the t_ν distribution with ν degrees of freedom, for each $\nu > 0$, so $\gamma = \Pr[-t^\gamma_\nu \le T \le t^\gamma_\nu]$ for a random variable T with the t_ν distribution.

7. How large a sample size n is required to ensure that a $\gamma = 0.99$ confidence interval for μ from iid variates $\{X_1, \ldots, X_n\} \overset{\text{iid}}{\sim} \mathsf{No}(\mu, 1)$ will have length less than 0.01?

8. With the same data as Problem 5, find a $p = 80\%$ posterior credible interval for μ if the variance is known to be $\sigma^2 = 16$, using an improper uniform prior distribution for μ.

XC: With the same data as Problem 5, find a $p = 80\%$ posterior credible interval for μ if $\sigma^2 = 1/\tau$ is unknown, using a normal-gamma prior distribution with a gamma distribution $\tau \sim \mathsf{Ga}(2, 20)$ for the precision $\tau = 1/\sigma^2$ and a normal conditional distribution $\mu|\tau \sim \mathsf{No}(25, 1/4\tau)$ for the mean μ.