1. For the normal linear model

\[ Y \sim N(X\beta, \mathbf{I}_n/\phi) \]  

(with \( X \) \( n \times p \) of rank \( p \)), consider the conjugate Normal-Gamma prior with

\[ \beta \mid \phi \sim N(\beta_0, (\phi \Phi)^{-1}) \]  

\[ \phi \sim \text{Gamma}(\nu_0/2, SS_0/2) \]

where \( \Phi \) is a symmetric positive definite matrix of rank \( p \). The conditional posterior for \( \beta \) is

\[ \beta \mid \phi, Y \sim N(\tilde{\beta}, (\phi (X^TX + \Phi))^{-1}) \]

where \( \tilde{\beta} = (X^TX + \Phi)^{-1}(X^TY + \Phi \beta_0) \) and \( \tilde{\beta} \) is the MLE. The posterior for \( \phi \) is Gamma with shape parameter \((n + \nu_0)/2\).

After completing the square, we showed that the rate parameter for the Gamma distribution for \( \phi \) was

\[ \left( \|Y - X\tilde{\beta}\|^2 + SS_0 + \beta_0^T \Phi \beta_0 + \tilde{\beta}^T (X^TX)\tilde{\beta} - \tilde{\beta}^T (X^TX + \Phi) \tilde{\beta} \right)/2. \]

Using the same priors, we used a projective method to derive the posterior by showing that the Normal prior may be viewed as “prior data” which led to rate parameter

\[ \left( \|Y - X\tilde{\beta}\|^2 + SS_0 + (\tilde{\beta} - \beta_0)^T \Phi (\tilde{\beta} - \beta_0) \right)/2. \]

Show that the two rate parameters are equal.

2. One version of what is called Zellner’s \( g \)-prior is based on taking \( \beta_0 = 0 \), and \( \Phi = X^TX/g \) for a fixed \( g \) and using a reference prior distribution for \( \phi \), \( p(\phi) \propto 1/\phi \). Can this be expressed as a limiting case of the conjugate prior distribution in (2) and (3)? If so, give the limiting values of \( \nu_0 \) and \( SS_0 \). Find the posterior distributions of \( \beta \mid \phi \) and \( \phi \). Show that the posterior distribution of \( \mu = X\beta \) given \( \phi \) may be expressed in terms of the projection matrix \( P_X = X(X^TX)^{-1}X \).

3. Let’s consider the problem of predicting swimming times for high school swimmers to swim 50 yard. Each student has six times taken on a biweekly basis (take biweekly to mean every two weeks rather than twice a week). A linear model for the swimming times \( T \) given the week is given by

\[ T_i = \beta_0 + \beta_1(W_i - \bar{W}) + \epsilon_i \]

where \( \epsilon_i \sim N(0, 1/\phi) \).
(a) Give an interpretation for all of the unknown parameters in the model.

(b) Show that the Fisher Information Matrix for this model is diagonal. This has led some to suggest that $\beta_0$ and $\beta_1$ in the above parameterization may be considered independent a priori.

(c) Using only the information that swimming times for this age group range from 22 to 24 seconds, construct an informative conjugate prior distribution for the parameters. Document how you construct your prior and give all the (final) values of the hyper-parameters.

(d) Generate realizations from the prior predictive distribution for a single swimmer over the 12 weeks and create a scatterplot of the predictive draws. Does this seem to agree with your prior beliefs? If not, explain why and adjust your prior hyper-parameters and repeat. In your write up, please show the sequence of “iterations” you go through if you have to adjust your prior distribution to agree with your beliefs.

(e) Using the data in the file swim.dat, find the OLS parameter estimates for each swimmer. Using this and your prior from above, give the posterior distributions $\beta_0$, $\beta_1$ and $\phi$ for each swimmer. Create credible intervals for the $\beta_0$ and $\beta_1$ and interpret.

(f) For each swimmer, plot their posterior predictive distributions for a future time $T_j^*$ two weeks after the last recorded observation (overlay the 4 densities in a single plot).

(g) The coach of the team has to recommend which of the swimmers to compete in a swim meet in two weeks time. Using draws from the predictive distributions, compute $P(T_j^* = \max(T_1^*, T_2^*, T_3^*, T_4^*))$ for each swimmer $j$, and based on this make a recommendation to the coach.