Martingales: Optional Sampling

Karlin & Taylor, *A First Course in Stochastic Processes, pp 253–278*

I. Review of Uniform Integrability
   A. Definitions
      1. $X_n$ are UI if $\forall\epsilon\exists a<\infty\forall n\ \int_{[|X_n|>a]}|X_n|\,dP<\epsilon$
      2. $X_n$ are UI if $\forall\epsilon\exists\delta\forall n\ \mathsf{P}[E]<\delta\Rightarrow\int_E|X_n|\,dP<\epsilon$
   B. Sufficient Conditions
      1. $X_n$ are UI if $\exists Y\in L^1$, $|X_n|\le Y$ *a.s.*
      2. $X_n$ are UI if $\exists p>1, M<\infty$, $\mathsf{E}|X_n|^p\le M$
      3. If $X_t$ is a martingale and $T<\infty$ then $\{X_t : t\le T\}$ is UI
   C. Consequences
      1. If $X_n\to X$ *in pr.* (or *a.s.*), then $X_n\to X$ in $L^1$ iff $\mathsf{E}|X_n|\to\mathsf{E}|X|$ iff the $X_n$ are UI.

II. Markov Times
   A. Definition: $[\omega:\tau(\omega)\le t]\in\mathcal{F}_t$
      1. Discrete time: $[\omega:\tau(\omega)=n]\in\mathcal{F}_n$ is good enough
      2. $[\omega:\tau(\omega)<n]\in\mathcal{F}_n$ is *not* good enough
   B. Examples
      1. Constant times: $\tau=t_0$
      2. Hitting times: $\tau=\inf[t\ge 0: X_t\in B]$ (especially, $\tau_R=\inf[t\ge 0:|X_t|>R]$; $X_{t\wedge\tau_R}$ is bounded)
      3. Jump times: $\tau=\inf[t\ge 0:|X_t-X_{t-}|>\epsilon]$ (including $\epsilon=0$)
      4. Successive hitting times: $\tau_n=\inf[t>\tau_{n-1}:X_t\in B]$
   C. Stability: If $\tau_n\ge 0$ are Markov times,
      1. Minimum: $\tau_1\wedge\tau_2$ is Markov
      2. Maximum: $\tau_1\vee\tau_2$ is Markov
      3. Sum: $\tau_1+\tau_2$ is Markov
      4. Difference: $\tau_1-\tau_2$ is **not** Markov
      5. Sup: $\sup_{n<\infty}\tau_n$ is Markov
      6. Approx: $\tau_n=\lfloor 2^n\tau\rfloor/2^n$ is a discrete Markov time, $\tau_n\le\tau$, and $\tau_n\nearrow\tau$ *a.s.*

III. Optional Sampling Theorem: If $\tau$ is a Markov time and $M_n$ a martingale, $\mathsf{E}[M_0]=\mathsf{E}[M_{\tau\wedge t}]$ for every $t<\infty$, and so $\mathsf{E}[M_0]=\lim_{t\to\infty}\mathsf{E}[M_{\tau\wedge t}]$. *Maybe* $\mathsf{E}[M_0]=\mathsf{E}[M_\tau]$.
   A. $\mathsf{E}[M_0]=\mathsf{E}[M_{\tau\wedge t}]$
      1. Easy for discrete $\tau$'s: $\sum_{t_i\le t}\int_{\tau=t_i}M_{t_i}\,dP+\int_{\tau>t}M_t\,dP=\int M_t\,dP=\mathsf{E}M_0$
      2. Now extend to all $\tau$'s, using I.B.3 and II.C.6
   B. $\mathsf{E}[M_0]=\mathsf{E}[M_\tau]$ *if* $\{M_t : t<\infty\}$ is UI:
      1. *e.g.*, if $Y=\sup_{t<\infty}|M_t|\in L^1$; or
      2. *e.g.*, if $\exists p>1\ni\sup_{t<\infty}\mathsf{E}|M_t|^p<\infty$
      3. *e.g.*, if $\mathsf{E}|M_\tau|<\infty$ and $\lim_{t\to\infty}\int_{[\tau>t]}|M_t|\,dP=0$
   C. Applications of Optional Sampling Theorem
      1. Wald's identity: Let $Y_i$ be iid with $\phi(\theta)=\mathsf{E}[e^{\theta Y_i}]$ satisfying $1<\phi(\theta)<\infty$ for some $\theta$; then if $S_n=\sum_{i\le n}Y_i$, then $X_n=e^{\theta S_n}\phi(\theta)^{-n}$ is a martingale. If $-a<0<b$ and $\tau=\inf[n:S_n<-a$ or $S_n>b]$, and if $\phi(\theta_0)=1$, then $1=\mathsf{E}X_0=\mathsf{E}X_{\theta S_\tau}\approx e^{\theta_0 b}\mathsf{P}[S_\tau\ge b]+e^{-\theta_0 a}\mathsf{P}[S_\tau\le -a]$, from which we can estimate $\mathsf{P}[S_\tau\ge b]$ and $\mathsf{E}[\tau]=\mathsf{E}[S_\tau]/\mathsf{E}[Y_1]$.
      2. Examples: Dams, option prices, markov chains

**Karlin & Taylor's Dam Example**

Let $Z_t$ be the reservoir water level on day $t$. A dam of height $b$ leads to the bound $Z_t \le b$; on day $t$, $I_t$ flows into the reservoir and (at most) $O_t$ flows out, so

$$Z_{t+1} = 0 \vee (Z_t + I_t - O_t) \wedge b.$$

Although we can control $b$ and $O_t$, $I_t$ is random and so $Z_t$ is a stochastic process. Suppose that it is important to have $Z_t > a$ for some $0 < a < b$; set $\tau \equiv \inf[t \ge 0 : Z_t < a]$. How should $b$ and $O_t$ be chosen to make sure $G(z) = \mathsf{E}[\tau \mid Z_0 = z]$ is large? How can we estimate $G(z)$?

Denote the net inflow by $Y_t = I_t - O_t$, so that $Z_{t+1}$ is just $Z_t + Y_t$, truncated to the interval $[0, b]$. Extend the definition of $G(z)$ so that $G(z) = 0$, $z \le a$, and $G(z) = G(b)$, $z \ge b$. Then $G(z)$ is increasing and satisfies, for $a < z < b$,

$$G(z) = \mathsf{E}[\tau \mid Z_0 = z] = 1 + \mathsf{E}[G(z + Y_t) \mid \mathcal{F}_t] \qquad \star$$

The stochastic process $X_t = G(Z_t) + t$ satisfies

$$X_t = G(Z_t) + t = 1 + \mathsf{E}[G(Z_t + Y_t) \mid \mathcal{F}_t] + t = \mathsf{E}[G(Z_{t+1}) + (t+1) \mid \mathcal{F}_t] = \mathsf{E}[X_{t+1} \mid \mathcal{F}_t],$$

*i.e.*, $X_t$ is a martingale. The strategy is to try to find a function $g(z)$ that nearly satisfies $\star$; to bound $\mathsf{E}[\tau]$ from below, we'll need a $g(z)$ satisfying

$$g(z) \le 1 + \mathsf{E}[g(z + Y_t) \mid \mathcal{F}_t] \qquad \star\star$$

which will make $X_t = g(Z_t) + t$ into a *sub*martingale, satisfying

$$X_t = g(Z_t) + t \le \mathsf{E}[X_{t+1} \mid \mathcal{F}_t] = \mathsf{E}[g(Z_{t+1}) + (t+1) \mid \mathcal{F}_t],$$

so *if* the optional sampling theorem applies, we can conclude

$$g(z) = \mathsf{E}[X_0 \mid Z_0 = z] \le \mathsf{E}[X_\tau \mid Z_0 = z] = \mathsf{E}[g(Z_\tau) + \tau \mid Z_0 = z] = \mathsf{E}[\tau \mid Z_0 = z]$$

and so that $\mathsf{E}[\tau \mid Z_0 = z] \ge g(z)$. The text verifies that, under stated conditions,

$$g(z) = \frac{1}{m}\Big[e^{\lambda(b-a)}\frac{1 - e^{-\lambda(z-a)}}{\lambda} - \big(z - a\big)\Big]$$

satisfies $\star\star$ and, therefore, $\mathsf{E}[\tau \mid Z_0 = z] \ge g(z)$.