Martingales: Optional Sampling


I. Review of Uniform Integrability

A. Definitions

1. $X_n$ are UI if $\forall \varepsilon > 0 \exists \eta > 0 \forall n \int_{[|X_n| > \varepsilon]} |X_n|dP < \epsilon$

2. $X_n$ are UI if $\forall \varepsilon > 0 \exists \eta > 0 \forall n \mathbb{P}[|E| < \delta \Rightarrow \int |X_n|dP < \epsilon]

B. Sufficient Conditions

1. $X_n$ are UI if $\exists Y \in L^1, |X_n| \leq Y$ a.s.

2. $X_n$ are UI if $\exists \rho > 1, M < \infty, E[|X_n|^\rho] \leq M$ for every $t < \infty$, and so $E[X_0] = \lim_{t \to \infty} E[X_t]$. Maybe $E[X_0] = E[X_t]$.

A. $E[X_0] = E[X_{\tau \wedge t}]$

1. Easy for discrete $\tau$: $\sum_{i \leq t} \int_{\tau = i} M_t dP + \int_{\tau > t} M_t dP = \int M_t dP = EM_0$

2. Now extend to all $\tau$’s, using I.B.3 and II.C.6

B. $E[X_0] = E[X_t]$ if $\{M_t : t < \infty\}$ is UI:

1. $e.g.$, if $Y = \sup_{t < \infty} |M_t| \in L^1$; or

2. $e.g.$, if $\exists \rho > 1 \exists \sup_{t < \infty} E[M_t^\rho] < \infty$

3. $e.g.$, if $E[M_t] < \infty$ and $\lim_{t \to \infty} \int_{|\tau| > \tau} |M_t|dP = 0$

C. Applications of Optional Sampling Theorem

1. Wald’s identity: Let $Y_i$ be iid with $\phi(\theta) = \mathbb{E}[e^{\theta Y_i}]$ satisfying $1 < \phi(\theta) < \infty$ for some $\theta$; then if $S_n = \sum_{i \leq n} Y_i$, then $X_n = e^{\theta S_n} - \phi(\theta)^{-n}$ is a martingale. If $-\alpha < b < \beta$ and $\tau = \inf\{n : S_n < -\alpha$ or $S_n > \beta\}$, and if $\phi(\theta_0) = 1$, then $1 = \mathbb{E}X_0 = \mathbb{E}X_{\tau \wedge t} \approx e^{\theta_0 \beta}P[S_\tau \geq \beta] + e^{\theta_0 \alpha}P[S_\tau \leq -\alpha]$, from which we can estimate $P[S_\tau \geq \beta]$ and $\mathbb{E}[\tau] = \mathbb{E}[S_\tau]/\mathbb{E}[Y_i]$.

2. Examples: Dams, option prices, markov chains
Karlin & Taylor’s Dam Example

Let $Z_t$ be the reservoir water level on day $t$. A dam of height $b$ leads to the bound $Z_t \leq b$; on day $t$, $I_t$ flows into the reservoir and (at most) $O_t$ flows out, so

$$Z_{t+1} = 0 \lor (Z_t + I_t - O_t) \land b.$$  

Although we can control $b$ and $O_t$, $I_t$ is random and so $Z_t$ is a stochastic process. Suppose that it is important to have $Z_t > a$ for some $0 < a < b$; let $\tau \equiv \inf\{t \geq 0 : Z_t < a\}$. How should $b$ and $O_t$ be chosen to make sure $G(z) = \mathbb{E}[\tau \mid Z_0 = z]$ is large? How can we estimate $G(z)$?

Denote the net inflow by $Y_t = I_t - O_t$, so that $Z_{t+1}$ is just $Z_t + Y_t$, truncated to the interval $[0, b]$. Extend the definition of $G(z)$ so that $G(z) = 0$, $z \leq a$, and $G(z) = G(b)$, $z \geq b$. Then $G(z)$ is increasing and satisfies, for $a < z < b$,

$$G(z) = \mathbb{E}[\tau \mid Z_0 = z] = 1 + \mathbb{E}[G(z + Y_t) \mid \mathcal{F}_t] \quad \star$$

The stochastic process $X_t = G(Z_t) + t$ satisfies

$$X_t = G(Z_t) + t = 1 + \mathbb{E}[G(Z_t + Y_t) \mid \mathcal{F}_t] + t = \mathbb{E}[G(Z_{t+1}) + (t + 1) \mid \mathcal{F}_t] = \mathbb{E}[X_{t+1} \mid \mathcal{F}_t],$$

i.e., $X_t$ is a martingale. The strategy is to try to find a function $g(z)$ that nearly satisfies $\star$; to bound $\mathbb{E}[\tau]$ from below, we’ll need a $g(z)$ satisfying

$$g(z) \leq 1 + \mathbb{E}[g(z + Y_t) \mid \mathcal{F}_t] \quad \star\star$$

which will make $X_t = g(Z_t) + t$ into a submartingale, satisfying

$$X_t = g(Z_t) + t \leq \mathbb{E}[X_{t+1} \mid \mathcal{F}_t] = \mathbb{E}[g(Z_{t+1}) + (t + 1) \mid \mathcal{F}_t],$$

so if the optional sampling theorem applies, we can conclude

$$g(z) = \mathbb{E}[X_0 \mid Z_0 = z] \leq \mathbb{E}[X_\tau \mid Z_0 = z] = \mathbb{E}[g(Z_\tau) + \tau \mid Z_0 = z] = \mathbb{E}[\tau \mid Z_0 = z]$$

and so that $\mathbb{E}[\tau \mid Z_0 = z] \geq g(z)$. The text verifies that, under stated conditions,

$$g(z) = \frac{1}{m} \left[ e^{\lambda(b-a)} \frac{1 - e^{-\lambda(z-a)}}{\lambda} - (z - a) \right]$$

satisfies $\star\star$ and, therefore, $\mathbb{E}[\tau \mid Z_0 = z] \geq g(z)$. 

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