BROWNIAN BRIDGE

S Karlin & J Taylor, A Second Course in Stochastic Processes, ch 15
G Kallianpur, Stochastic Filtering Theory, ch 5

Let \( w(t) \) be a standard Brownian Motion, i.e., a continuous-path Gaussian stochastic process with mean \( \mathbb{E}[w(t)] \equiv 0 \) and covariance \( \mathbb{E}[w(s)w(t)] = (s \wedge t) \), the minimum of \( s \) and \( t \) (for nonnegative \( s, t \)). Now define two processes for \( 0 \leq t \leq 1 \) by:

\[
X(t) \equiv w(t) - tw(1)
\]

\[
Y(t) \equiv (1-t)w \left( \frac{t}{1-t} \right)
\]

(1)

(2)

It is easy to see that both \( X(\cdot) \) and \( Y(\cdot) \) are continuous-path mean-zero Gaussian stochastic processes (because \( w(\cdot) \) is); their covariances for \( 0 \leq s \leq t \leq 1 \) are:

\[
\mathbb{E}[X(s)X(t)] = \mathbb{E} \left[ (w(s) - sw(1)) \left( w(t) - tw(1) \right) \right]
\]

\[
= s - st - st + st
\]

\[
= (s \wedge t) - st
\]

(3a)

\[
\mathbb{E}[Y(s)Y(t)] = (1-s)(1-t) \left( \frac{s}{1-s} \wedge \frac{t}{1-t} \right)
\]

\[
= (1-t)s
\]

\[
= (s \wedge t) - st
\]

(3b)

Each of these is the **Brownian Bridge**, a zero-mean continuous-path Gaussian process with covariance \( (s \wedge t) - st \).

**Properties**

For \( 0 \leq t \leq 1 \) let \( \mathcal{F}_t = \sigma \{ Y_s : s \leq t \} \) be the \( \sigma \)-algebra generated by the Brownian Bridge; from (2) it is clear that, for \( t < 1 \) and \( 0 < \epsilon < 1 - t \), the conditional distribution of \( Y(t + \epsilon) \), given the past \( \mathcal{F}_t \), is Gaussian with nonzero conditional predictive mean

\[
\mathbb{E}[Y(t + \epsilon)|\mathcal{F}_t] = \mathbb{E} \left[ (1-t-\epsilon)w \left( \frac{t}{1-t} \right) + w \left( \frac{t+\epsilon}{1-t-\epsilon} \right) - w \left( \frac{t}{1-t} \right) \right] \bigg| \mathcal{F}_t
\]

\[
= (1-t-\epsilon)w \left( \frac{t}{1-t} \right)
\]

\[
= \frac{1-t-\epsilon}{1-t} Y(t) = Y_t - \epsilon \frac{Y_t}{1-t}
\]

(4)

and conditional predictive variance

\[
\mathbb{E}[(Y(t + \epsilon) - \mathbb{E}[Y(t + \epsilon)|\mathcal{F}_t])^2 | \mathcal{F}_t] = \mathbb{E} \left[ (1-t-\epsilon)^2 \left( w \left( \frac{t+\epsilon}{1-t-\epsilon} \right) - w \left( \frac{t}{1-t} \right) \right)^2 \right] | \mathcal{F}_t
\]

\[
= (1-t-\epsilon)^2 \left( \frac{t+\epsilon}{1-t-\epsilon} - \frac{t}{1-t} \right)
\]

\[
= (1-t-\epsilon)^2 \left( \frac{(t+\epsilon)(1-t) - t(1-t-\epsilon)}{(1-t)(1-t-\epsilon)} \right)
\]

\[
= \frac{\epsilon(1-t-\epsilon)}{1-t} = \epsilon - \epsilon^2 \frac{1}{1-t}
\]

(5)
By equation (4) we see that $Y(t)$ is not a martingale, but that

$$W_t \equiv Y(t) + \int_0^t \frac{1}{1-s} Y(s) \, ds$$

is a martingale. In fact $W_t$ is a continuous-path Gaussian martingale, starting at $W_0 = 0$, with mean zero. From the martingale property we must have $\mathbb{E}[W_t W_t] = \mathbb{E}[W_s^2]t$ (since, for $s < t$, $\mathbb{E}[W_s(W_t - W_s)|\mathcal{F}_s] = 0$), while the variance is given by

$$\begin{align*}
\mathbb{E}[W_t^2] &= t(1-t) + 2 \int_0^t \frac{u - u t}{1-u} \, du + 2 \int_0^t \int_0^t \frac{u - u v}{(1-u)(1-v)} \, du \, dv \\
&= t - t^2 + 2 \int_0^t \frac{u(1-t) + u(t-u)}{1-u} \, du \\
&= t, \quad \text{so} \\
\mathbb{E}[W_s W_t] &= s \wedge t.
\end{align*}$$

Thus we see that $W(t)$ is just a Wiener process and we have the (third!) representation of $Y(t)$ as the solution to a Stochastic Integral Equation:

$$Y(t) = 0 - \int_0^t \frac{1}{1-s} Y(s) \, ds + W_t, \quad (6)$$

from which it is again apparent that the conditional mean and variance of $Y(t + \epsilon)$, given $\mathcal{F}_t$, are (respectively)

$$\begin{align*}
\mathbb{E}[Y(t + \epsilon)|\mathcal{F}_t] &= Y(t) - \epsilon \frac{Y(t)}{1-t} + \sigma(\epsilon) \\
\mathbb{V}[Y(t + \epsilon)|\mathcal{F}_t] &= \epsilon + \sigma(\epsilon),
\end{align*}$$

just as in equations (4) and (5). This is our first example of a Diffusion; it is common to write this in differential form as

$$dY(t) = - \frac{1}{1-t} Y(t) \, dt + dW_t,$$

and to write more general diffusions in the similar form

$$dX_t = \mu(t, X_t) \, dt + \sigma(t, X_t) \, dW_t; \quad \text{(SDE)}$$

it remains to make sense of the stochastic integral in the integral form of this equation,

$$X_t = x_0 + \int_0^t \mu(s, X_s) \, ds + \int_0^t \sigma(s, X_s) \, dW_s. \quad \text{(SIE)}$$

The Brownian Bridge, our first example, had $x_0 = 0$, $\mu(s, x) = -x/(1-s)$, and $\sigma(s, x) = 1$; the Wiener process (scaled Brownian motion with drift) could be written in the same form with constant $\mu(s, x) \equiv \mu$ and $\sigma(s, x) \equiv \sigma$, while standard Brownian motion itself has simply $x_0 = 0$, $\mu(s, x) \equiv 0$, and $\sigma(s, x) \equiv 1$. Our next diffusion, the Ornstein-Uhlenbeck velocity (OUV) process, is the continuous-time analogue of an AR(1) time-series, with $\mu(s, x) = -\beta x$ and $\sigma(s, x) \equiv 1$, so its SIE representation will be:

$$X_t = x_0 - \beta \int_0^t X_s \, ds + W_t.$$
Lemma (Gronwall’s Inequality). Suppose for \( t > 0 \) the nonnegative function \( f(t) \geq 0 \) satisfies the inequality

\[
f(t) \leq \alpha_t + \beta \int_0^t f(s) \, ds.
\]

Then \( f(t) \leq \alpha_t + \beta \int_0^t e^{\beta(t-s)} \alpha_s \, ds \) for all \( t > 0 \). In particular, if \( \alpha_s \equiv \alpha \), then \( f(t) \leq \alpha e^{\beta t} \), and if \( \alpha_s \equiv 0 \), then \( f(t) \leq \beta \int_0^t f(s) \, ds \) entails \( f(t) \equiv 0 \).

Proof. Iterate:

\[
f(t) \leq \alpha_t + \beta \int_0^t f(s) \, ds
\]

\[
\leq \alpha_t + \beta \int_0^t \left( \alpha_s + \beta \int_0^s f(s) \, ds \right) \, ds'
\]

\[
= \alpha_t + \beta \int_0^t (1 + \beta(t-s)) \alpha_s \, ds + \beta \int_0^t \beta(t-s)f(s) \, ds
\]

\[
\leq \alpha_t + \beta \int_0^t \left( 1 + \frac{\beta^2(t-s)^2}{2!} \right) \alpha_s \, ds + \beta \int_0^t \frac{\beta^n(t-s)^n}{n!} f(s) \, ds
\]

\[
\leq \alpha_t + \beta \int_0^t e^{\beta(t-s)} \alpha_s \, ds.
\]

\[\square\]

Theorem (SDE). Suppose for \( t > 0 \) and \( x \in \mathbb{R} \) the function \( \mu(t,x) \in \mathbb{R} \) satisfies the Lipschitz inequality and linear growth condition

\[
|\mu(s,x) - \mu(s,y)| \leq \beta |x - y| \quad |\mu(s,x)|^2 \leq \beta \left( 1 + |x|^2 \right).
\]

Then for any continuous function \( w : \mathbb{R}_+ \to \mathbb{R} \) and numbers \( x_0 \in \mathbb{R}, \sigma > 0 \), there is a unique solution to the integral equation

\[
X_t = x_0 + \int_0^t \mu(s,X_s) \, ds + \sigma w_t.
\]

Proof. For any continuous \( f \in C(\mathbb{R}_+) \) define a new element \( Tf \in C(\mathbb{R}_+) \) by:

\[
Tf(t) \equiv x_0 + \int_0^t \mu(s,f(s)) \, ds + \sigma w_t.
\]

Starting with \( X^{(0)} \equiv x_0 \), define a succession of approximations to \( X_t \) by

\[
X^{(n)}(t) \equiv TX^{(n-1)}(t) \equiv x_0 + \int_0^t \mu(s,X^{(n-1)}(s)) \, ds + \sigma w_t.
\]

Then \( f^{(n)}(t) \equiv |X^{(n+1)}(t) - X^{(n)}(t)| \) satisfies

\[
f^{(n)}(t) \leq \int_0^t |\mu(s,X^{(n)}) - \mu(s,X^{(n-1)})| \, ds
\]

\[
\leq \int_0^t \beta |X^{(n)} - X^{(n-1)}| \, ds = \beta \int_0^t f^{(n-1)}(s) \, ds.
\]
so we can mimic the Gronwall proof:

\[
f^{(n)}(t) \leq \beta \int_0^t f^{(n-1)}(s) \, ds \leq \beta \int_0^t \int_0^t \beta f^{(n-2)}(s) \, ds \, ds' \\
= \beta^2 \int_0^t (t-s) f^{(n-2)}(s) \, ds \leq \beta^2 \int_0^t \int_0^t (t-s') \beta f^{(n-3)}(s) \, ds \, ds' \\
= \beta^3 \int_0^t \frac{(t-s)^2}{2!} f^{(n-3)}(s) \, ds \leq \cdots \leq \\
\leq \beta^n \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f^{(0)}(s) \, ds \\
\to 0
\]

as \( n \to \infty \) by Lebesgue’s dominated convergence theorem, since \( \sum \frac{\beta^{n+1}(t-s)^n}{(n-1)!} = \beta e^{\beta(t-s)} < \infty \).