Sta 250 = Mth 342 : Homework 6

Problems

1. Find the limit as $\nu \to \infty$ of the un-normalized t density,

$$\frac{1}{(1 + x^2/\nu)^{(\nu+1)/2}}$$

   Do you recognize it?

2. The t distribution was introduced (both by our text book authors and, early last century, by William Gossett who invented it and published about it under the pseudonym *Student*) as a ratio

$$t = \frac{Z}{\sqrt{Y/\nu}}$$

   for independent random variables $Z \sim \mathsf{No}(0, 1)$ and $Y \sim \chi^2_\nu$. Show that t^2 is the ratio of two independent Gamma-distributed random variables, each with mean one. Find the parameters for each Gamma distribution.

3. Let $\{X_1, \dots, X_n\} \overset{\text{iid}}{\sim} \mathsf{No}(\mu, \sigma^2)$ be a simple random sample from the Normal distribution with unknown mean μ but **known** variance σ^2. Denote the standard normal CDF by $\Phi(z)$. For any $0 < \gamma < 1$ let z^*_γ be the $(1 + q)/2$'th quantile of the $\mathsf{No}(0, 1)$ distribution, so $\Phi(z^*_\gamma) = \frac{1+\gamma}{2}$ and $\Phi(-z^*_\gamma) = \frac{1-\gamma}{2}$. If $\bar{x}_n$ denotes the sample mean of $\{X_i\}$, show that

$$\left(\bar{x}_n - \frac{z^*_\gamma \, \sigma}{\sqrt{n}}, \quad \bar{x}_n + \frac{z^*_\gamma \, \sigma}{\sqrt{n}}\right)$$

   is a confidence interval for μ with coefficient γ.

4. How large a sample size n is required to ensure that a $\gamma = 0.99$ confidence interval for μ from iid variates $\{X_1, \dots, X_n\} \overset{\text{iid}}{\sim} \mathsf{No}(\mu, 1)$ will have length less than 0.01?

5. The eight observations $\{31, 35, 26, 34, 38, 30, 29, 25\}$ are taken from a normal $\mathsf{No}(\mu, \sigma^2)$ distribution with both μ and σ unknown. Find the shortest confidence interval for μ with coefficient $\gamma = 0.80$ and (separately) with $\gamma = 0.98$.

6. With the same data as Problem 5, find a $p = 80\%$ posterior credible interval for μ if the variance is known to be $\sigma^2 = 16$, using an improper uniform prior distribution for μ.

XC: With the same data as Problem 5, find a $p = 80\%$ posterior credible interval for μ if $\sigma^2 = 1/\tau$ is unknown, using a normal-gamma prior distribution with a gamma distribution $\tau \sim \mathsf{Ga}(2, 20)$ for the precision $\tau = 1/\sigma^2$ and a normal conditional distribution $\mu|\tau \sim \mathsf{No}(25, 1/4\tau)$ for the mean μ.