Midterm Examination I

STA 532: Statistical Inference

Friday, 2015 Feb 13, 8:30 – 9:45am

This is a closed-book exam. You may use a single sheet of prepared notes, if you wish, but you may not share materials.

If a question seems ambiguous or confusing, *please* ask me to clarify it. Unless a problem states otherwise, you must **show** your **work**. There are blank worksheets at the end of the test if you need more room for this.

It is to your advantage to write your solutions as clearly as possible, and to $\boxed{\text{box}}$ answers I might not find.

For full credit, give answers in **closed form** (no unevaluated sums, integrals, maxima, unreduced fractions, *etc.*) where possible and **simplify**.

Good luck!

Print Name Clearly: ______________________________

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
Total:	/100

Problem 1: The following statistics are from n observations $X_i \overset{\text{iid}}{\sim} \mathsf{Be}(\theta, 1)$ from the Beta distribution, with parameter $\beta = 1$ known, hence pdf $f(x \mid \theta) = \theta x^{\theta-1} \mathbf{1}_{\{0<x<1\}}$:

$$\begin{array}{l|l|l} \sum X_i = A & \sum X_i^2 = B & \sum X_i^4 = C \\ \sum \log X_i = D & \sum \log(1 - X_i) = E & \sum 1/X_i = F \end{array}$$

Give answers as expressions in n, A, B, C, D, E, and/or F.

a) Find the Method of Moments estimate: $\tilde{\theta} =$ ______________

b) Find the Maximum Likelihood estimate: $\hat{\theta} =$ ______________

c) Find the Maximum Likelihood estimate for the mean $\mu := \mathsf{E}X_i$ and variance $\sigma^2 := \mathsf{E}|X_i - \mu|^2$:

$\hat{\mu} =$ ______________ $\hat{\sigma}^2 =$ ______________

Problem 1 (cont'd): The following summary statistics are (still) from n observations $X_i \overset{\text{iid}}{\sim} \mathsf{Be}(\theta, 1)$ from the Beta distribution:

$$\begin{array}{l|l|l} \sum X_i \quad = A & \sum X_i^2 \qquad\qquad = B & \sum X_i^4 \quad = C \\ \sum \log X_i = D & \sum \log(1 - X_i) = E & \sum 1/X_i = F \end{array}$$

d) Let $\theta \sim \mathsf{Ga}(\alpha, \lambda)$ have a Gamma prior probability distribution. Find the posterior distribution for θ, given the n observations described here (by name with parameter values), and give the Bayesian estimator that minimizes squared-error loss:

$\pi(\theta \mid \mathbf{x}) \sim$ ______________ $\qquad \bar{\theta}(\mathbf{x}) =$ ______________

e) Find the Fisher information for n observations $\mathbf{x} = \{X_i\} \overset{\text{iid}}{\sim} \mathsf{Be}(\theta, 1)$: $I_n(\theta) =$

f) Find the Jeffreys' Rule (reference) prior density and estimator:

$\pi_R(\theta) =$ ______________ $\qquad \bar{\theta}_R(\mathbf{x}) =$ ______________

g) Find the Cramér-Rao (information inequality) lower bound for the MSE of an estimator with mean $m(\theta) := \mathsf{E}_\theta T(\mathbf{x})$ and bias $\beta(\theta) := [m(\theta) - \theta]$:

$$\mathsf{E}_\theta |T(\mathbf{x}) - \theta|^2 \geq \text{______________}$$

Problem 2: The times-to-failure (in days) T_j of electronic components called quixars are independent random variables with pdf's

$$f(x \mid \lambda) = \lambda^2 x e^{-\lambda x}, \qquad x > 0$$

for some unknown $\lambda > 0$, for $1 \le j \le n$. We have $n = 4$ observations, with failure-times $T_1 = 2$, $T_2 = 3$, $T_3 = 5$, $T_4 = 6$.

a) (5) For these data, find the Likelihood Function $f_n(x \mid \lambda)$:

b) (5) Find the maximum likelihood estimate $\hat{\lambda}_n(x)$:

c) (5) With an improper uniform prior $\pi(\lambda) \equiv \mathbf{1}_{\{\lambda>0\}}$, find the posterior density function $\pi(\lambda \mid x)$ (or the name of the posterior distribution, with the value(s) of any parameter(s)):

d) (5) Give the posterior mean $\mathsf{E}[\lambda \mid x]$ for this prior (Reminder: a table of familiar pdf's is attached to this test).

Problem 3: Twenty-five measurements X_j come from a normal distribution with mean $\mathsf{E}[X_j] = \mu$ and variance $\mathsf{E}[(X_j - \mu)^2] = 9$. They satisfy:

$$\sum_{j=1}^{25} X_j = 250 \qquad\qquad \sum_{j=1}^{25} X_j^2 = 3725$$

a) (4) Find the Maximum Likelihood Estimator for μ (you don't have to prove it's the MLE, just show how you found its value):

$\hat{\mu} =$ ______________________

b) (8) Find a (frequentist) 90% Confidence Interval for μ, *i.e.*, an interval with $\mathsf{P}_\mu[L(X) < \mu < R(X)] = 0.90$:

$L(X) =$ ______________________ $R(X) =$ ______________________

c) (8) Find a (Bayesian) 90% Credible Interval for μ, *i.e.*, an interval with $\mathsf{P}[L(X) < \mu < R(X) \mid X] = 0.90$, using the Reference (or Jeffreys' Rule) prior density $\pi_R(\mu)$:

$L(X) =$ ______________________ $R(X) =$ ______________________

Warning: Sometimes you have more information than you need...

Problem 4: For $N = 625$ road segments we have recorded the length X_j (in kilometers) and a count Y_j of pavement defects. We model $\{Y_j\}$ as independent Poisson random variables with conditional mean $\mathsf{E}[Y_j \mid X_j] = \theta X_j$ for some uncertain parameter $\theta > 0$, the average defect rate (per kilometer).

a) (4) Find the likelihood function for θ:

$L(\theta)$ __________________

b) (8) The MLE $\hat{\theta}$ will have approximately a Normal distribution $\mathsf{No}(\mu, \sigma^2)$ with some mean μ and variance σ^2. Give the MLE and its mean and variance:

$\hat{\theta} =$ ______________ $\mu =$ ______________ $\sigma^2 =$ ______________

c) (8) The MLE $\hat{\eta}$ for the *log* rate $\eta = \log \theta$ will also have approximately a Normal distribution $\mathsf{No}(\nu, \tau^2)$, with some mean ν and some variance τ^2. Give the MLE and its approximate mean and variance:

$\hat{\eta} =$ ______________ $\nu =$ ______________ $\tau^2 =$ ______________

Problem 5: Stacy and Toby each discovered a possible cure for cancer. Let θ_S and θ_T be the population fraction for which Stacy's and Toby's treatments are effective, respectively. In a randomized clinical trial (RCT), $N_S = 25$ subjects are given Stacy's treatment (of whom $S = 10$ show marked improvement) and $N_T = 30$ subjects are given Toby's treatment (of whom $T = 20$ show marked improvement).

Our statistician Chris elects to use independent uniform prior distributions $\pi(\theta_S) \equiv 1,\ 0 < \theta_S < 1$ and $\pi(\theta_T) \equiv 1,\ 0 < \theta_T < 1$.

a) (8) What are the posterior distributions for θ_S and θ_T? Give the pdfs **or** (better) the name and value(s) of any parameter(s).

$\theta_S \mid S \sim$ ________________ $\theta_T \mid T \sim$ ________________

b) (8) What are the posterior means and variances for the two treatments? Remember the attached pdf sheet...

$\mathsf{E}[\theta_S \mid S = 10] =$ _________ $\mathsf{V}[\theta_S \mid S = 10] =$ _________

$\mathsf{E}[\theta_T \mid T = 20] =$ _________ $\mathsf{V}[\theta_T \mid T = 20] =$ _________

c) (4) If the posterior distributions of θ_S and θ_T are well enough approximated by normal distributions, what would be the approximate probability $\mathsf{P}[\theta_S > \theta_T]$? Why?

$\mathsf{P}[\theta_S > \theta_T \mid S, T] \approx$ ____________________________

Extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

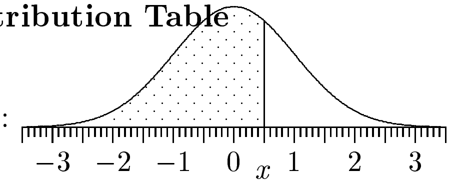


Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

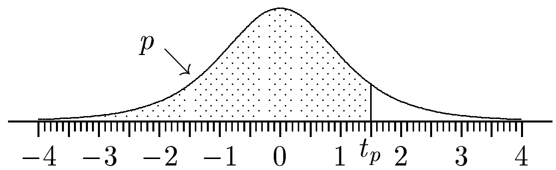
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Critical Values for Student's t

$$p = \int_{-\infty}^{t_p} c \, \frac{dt}{(1 + t^2/\nu)^{(\nu+1)/2}}$$



ν	$t_{.60}$	$t_{.70}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$	$t_{.9999}$
1	0.325	0.727	1.376	1.9626	3.078	6.314	12.76	31.82	63.66	318.3	636.6	3183.
2	0.289	0.617	1.061	1.3862	1.886	2.920	4.303	6.965	9.925	22.33	31.60	70.70
3	0.277	0.584	0.978	1.2498	1.638	2.353	3.182	4.541	5.841	10.22	12.92	22.20
4	0.271	0.569	0.941	1.1896	1.533	2.132	2.776	3.747	4.604	7.173	8.610	13.03
5	0.267	0.559	0.920	1.1558	1.476	2.015	2.571	3.365	4.032	5.893	6.869	9.678
6	0.265	0.553	0.906	1.1342	1.440	1.943	2.447	3.143	3.707	5.208	5.959	8.025
7	0.263	0.549	0.896	1.1192	1.415	1.895	2.365	2.998	3.499	4.785	5.408	7.063
8	0.262	0.546	0.889	1.1081	1.397	1.860	2.306	2.896	3.355	4.501	5.041	6.442
9	0.261	0.543	0.883	1.0997	1.383	1.833	2.262	2.821	3.250	4.297	4.781	6.010
10	0.260	0.542	0.879	1.0931	1.372	1.812	2.228	2.764	3.169	4.144	4.587	5.694
11	0.260	0.540	0.876	1.0877	1.363	1.796	2.201	2.718	3.106	4.025	4.437	5.453
12	0.259	0.539	0.873	1.0832	1.356	1.782	2.179	2.681	3.055	3.930	4.318	5.263
13	0.259	0.538	0.870	1.0795	1.350	1.771	2.160	2.650	3.012	3.852	4.221	5.111
14	0.258	0.537	0.868	1.0763	1.345	1.761	2.145	2.624	2.977	3.787	4.140	4.985
15	0.258	0.536	0.866	1.0735	1.341	1.753	2.131	2.602	2.947	3.733	4.073	4.880
16	0.258	0.535	0.865	1.0711	1.337	1.746	2.120	2.583	2.921	3.686	4.015	4.791
17	0.257	0.534	0.863	1.0690	1.333	1.740	2.110	2.567	2.898	3.646	3.965	4.714
18	0.257	0.534	0.862	1.0672	1.330	1.734	2.101	2.552	2.878	3.610	3.922	4.648
19	0.257	0.533	0.861	1.0655	1.328	1.729	2.093	2.539	2.861	3.579	3.883	4.590
20	0.257	0.533	0.860	1.0640	1.325	1.725	2.086	2.528	2.845	3.552	3.85	4.539
21	0.257	0.532	0.859	1.0627	1.323	1.721	2.080	2.518	2.831	3.527	3.819	4.493
22	0.256	0.532	0.858	1.0614	1.321	1.717	2.074	2.508	2.819	3.505	3.792	4.452
23	0.256	0.532	0.858	1.0603	1.319	1.714	2.069	2.500	2.807	3.485	3.768	4.415
24	0.256	0.531	0.857	1.0593	1.318	1.711	2.064	2.492	2.797	3.467	3.745	4.382
25	0.256	0.531	0.856	1.0584	1.316	1.708	2.060	2.485	2.787	3.450	3.725	4.352
26	0.256	0.531	0.856	1.0575	1.315	1.706	2.056	2.479	2.779	3.435	3.707	4.324
27	0.256	0.531	0.855	1.0567	1.314	1.703	2.052	2.473	2.771	3.421	3.690	4.299
28	0.256	0.530	0.855	1.0560	1.313	1.701	2.048	2.467	2.763	3.408	3.674	4.275
29	0.256	0.530	0.854	1.0553	1.311	1.699	2.045	2.462	2.756	3.396	3.659	4.254
30	0.256	0.530	0.854	1.0547	1.310	1.697	2.042	2.457	2.750	3.385	3.646	4.234
40	0.255	0.529	0.851	1.0500	1.303	1.684	2.021	2.423	2.704	3.307	3.551	4.094
60	0.254	0.527	0.848	1.0455	1.296	1.671	2.000	2.390	2.660	3.232	3.460	3.962
120	0.254	0.526	0.845	1.0409	1.289	1.658	1.980	2.358	2.617	3.160	3.373	3.837
∞	0.253	0.524	0.842	1.0364	1.282	1.645	1.960	2.326	2.576	3.090	3.291	3.719

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,\ldots\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\,\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,\ldots\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/(x+\epsilon)^{\alpha+1}$	$x\in\mathbb{R}_+$	$\frac{\epsilon}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
		$f(y)=\alpha\,\epsilon^\alpha/y^{\alpha+1}$	$y\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	$(y=x+\epsilon)$
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	