Sta 532: Homework #9 [Revised]

For all three problems, let \( \{X_i : 1 \leq i \leq n\} \overset{iid}{\sim} \text{Ex}(\theta) \).

**Problem 1:** Consider the two possible hypotheses

\[ H_0 : \theta = \theta_0, \quad H_1 : \theta = \theta_1. \]

Find the posterior probability of \( H_0 \) if the prior probabilities are \( P[\theta = \theta_0] = p \in (0, 1) \), \( P[\theta = \theta_1] = q = (1 - p) \).

**Problem 2:** Now consider the hypotheses

\[ H_0 : \theta = \theta_0, \quad H_1 : \theta \neq \theta_0 \]

with prior distribution depending on three parameters \( p \in (0, 1), \alpha > 0, \beta > 0 \) given by

\[
P[\theta \in A] = p \mathbf{1}_A(\theta_0) + q \int_{A} \theta^{\alpha-1} e^{-\beta \theta} / \Gamma(\alpha) \ d\theta, \quad A \subset \mathbb{R}_+.
\]

This is a mixture model, with a point mass of size \( p \) at \( \theta_0 \) and the rest of the prior mass \( q = (1 - p) \) distributed as \( \text{Ga}(\alpha, \beta) \).

a) Find the posterior odds \( P[H_0 | X] / P[H_1 | X] \). Simplify as much as possible.

b) The Jeffreys Prior distribution for the exponential distribution is \( \pi_J(\theta) \propto \theta^{-1} \), the limiting case of \( \text{Ga}(\alpha, \beta) \) as \( \alpha \to 0 \) and \( \beta \to 0 \). Find the limiting posterior odds.

**Problem 3:** One way to cheat at dice games is to shave the faces showing 1 and 6 a little, so that the other four faces will be a little smaller. This causes 1 and 6 to occur with higher frequency, leading to more rolls of “7” than would happen with fair dice.

Let \( p_j \) denote the probability of face \( j \) showing for a particular die which may or may not be fair. On the basis of these counts:

<table>
<thead>
<tr>
<th>Face:</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Count</td>
<td>12</td>
<td>7</td>
<td>9</td>
<td>13</td>
<td>5</td>
<td>14</td>
</tr>
</tbody>
</table>

consider the two hypotheses:

\[ H_0 : \vec{p} = [\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}] \quad vs. \quad H_1 : \vec{p} = [\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}] \]

- Find the likelihood ratio \( \Lambda \) against \( H_0 \).
- Find the posterior probability of \( H_0 \), with prior probability \( \pi_0 = 0.9 \) of a fair die.
- Find the value of the \( \chi^2 \) statistic \( Q \), and the \( P \)-value for the \( \chi^2 \) test of \( H_0 \).
- Which of these two tests is more sensitive (or powerful) at detecting this particular form of cheating at dice? Why?
Problem 4: [Optional, XC] Consider a sequential test of the hypotheses

\[ H_0 : \theta = \theta_0, \quad H_1 : \theta = \theta_1 \]

for the specific values \( \theta_0 = 2 \) and \( \theta_1 = 3 \).

a) Find the lower and upper limits \( a \) and \( b \) for \( \Lambda_n \) for a SLRT with approximate error probabilities \( \alpha \approx 0.01 \) and \( \beta \approx 0.02 \).

b) Describe the test precisely in terms of \( \bar{X}_n \) — i.e., find the numbers \( L_n \) and \( R_n \) such that the test ends when \( \bar{X}_n \leq L_n \) or \( \bar{X}_n \geq R_n \), and in each case say whether to Reject \( H_0 \) or not. Simplify and be explicit.

c) Find the approximate expected sample size under both \( H_0 \) and \( H_1 \). Give answer numerically (but also show your work so we’ll know how you got it).

d) This SPRT may also be viewed as a sequential test of \( H_0 \) against the one-sided composite test \( H_1' : \theta > \theta_0 \). Find its approximate power if in fact \( \theta = 2.75 \) — i.e., find the probability that \( H_0 \) will be rejected. Hint: Find \( p \in \mathbb{R} \) s.t. \( \Lambda_n^p \) is a martingale if \( \theta = 2.75 \). Also find the approximate expected sample size needed if \( \theta = 2.75 \).