Let $\mathcal{H}_s$ be the Hilbert space completion of $C_c^\infty(0,1)$ in the inner product
\[ \langle f, g \rangle_s = \sum_{n \in \mathbb{N}} (\pi^2 n^2)^s a_n b_n, \]
where $a_n := \int f(s) \psi_n(s) \, ds$ and $b_n := \int g(s) \psi_n(s) \, ds$ (all integrals are over $[0,1]$) for
\[ \psi_n(s) := \sqrt{2} \sin(n \pi s) \]
and, for $0 < s, t < 1$, set
\[ \gamma(s,t) := s \wedge t - st. \]

1. For which $s \in \mathbb{R}$ is $f(x) := x(1-x)$ in $\mathcal{H}_s$? Find $\|f\|_1^2$.

2. Let $B_t \sim \text{No}(0,\gamma())$ be the Brownian Bridge. Is $B_s$ a Markov process? Compute the conditional expectations
\[ \mathbb{E}[B_u \mid B_s = x, B_t = y] \quad \text{and} \quad \mathbb{E}[B_u \mid B_t = y] \]
for $0 < s < t < u < 1$ (if they differ then $B_s$ can’t be Markov). If this doesn’t show that $B_s$ is not Markov, try to use one of the representations of $B(t)$ in Section 2 of class lecture notes “Dirichlet Sobolev Spaces on $[0,1]$ and the Brownian Bridge” to prove that it is.

3. Let $W$ be the unit Gaussian process on $\mathcal{H}_0$, i.e., an isomorphism
\[ W : \mathcal{H}_0 \to L_2(\Omega, \mathcal{F}, \mathbb{P}) \]
that maps elements $\phi \in \mathcal{H}_0$ to Gaussian random variables $W[\phi] \sim \text{No}(0,\|\phi\|_0^2)$ such that $\text{Cov}(W[\phi], W[\psi]) = \langle \phi, \psi \rangle_0 = \int \phi(s) \psi(s) \, ds$. For $0 \leq t \leq 1$ set $g_t(s) := 1_{s \leq t}$ and $X_t := W[g_t]$. Find the mean, variance, and covariance functions for $X_t$ explicitly.

4. The Gaussian process $Y_s$ on the unit interval with mean zero and covariance function
\[ h(s,t) = \frac{1}{2}(s-t)^2 - \frac{1}{2}|s-t| + 1/12 \]
is “periodic Brownian motion,” with paths that are periodic functions on $[0,1]$. Show that the sample paths are orthogonal to the constants in $\mathcal{H}_0$—i.e., that $\int Y_s \, ds = 0$ almost surely. (Hint: Set $Z := \int Y_s \, ds$ and consider $\text{Cov}(Y_s, Z)$ or $\mathcal{V}(Z)$).

5. What is the generator $\mathcal{G}$ for the periodic Brownian Motion process $Y_s$ above? Find $\mathcal{G}$ such that
\[ \mathbb{E} Y[\phi] Y[\mathcal{G}\psi] = \langle \phi, \psi \rangle_0 \]
(Hint: $\mathcal{G}$ will be the inverse of the integral operator that takes $\psi$ to $g(s) = \int h(s,t) \psi(t) \, dt$. Try taking a derivative or two of $g$ and try to find $\mathcal{G}$ s.t. $\mathcal{G}g = \psi$). Find the domain $\mathcal{D}(\mathcal{G})$ and the value of $\mathcal{G}g$ for $g \in \mathcal{G}(\mathcal{G})$. 

Last edited: January 21, 2016