I solutions

Note: Version B had slightly different numbers. But the basic problems were the same. You can recognize Version B by “Name” (instead of “Name:”) on the top line, i.e., a missing “:” after “Name”.

1. On questions 1a-f: [2pts] for the correct choice; [0pts] for no choice; [-1pt] for a wrong choice. If more than one choice is correct, any correct choice is fine.

1a.

\[ P(A \cap B \cap C) = P(A|D)P(D) = P(A|B \cap C)P(B|C)P(C) = P(D) = P(B \cap C) \]

1b. By Bayes’ theorem:

\[ P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)[1 - P(A)]} \]

1c & d. Consider the following histogram of \( n = 300 \) measurements:

Denote with \( \bar{x} \) the sample mean, with \( Md \) the sample median, and with \( s^2 \) the sample variance. Which of the following statements is correct?

\[ \bar{x} > Md \text{ and } s^2 < 4. \]

1e. Peter is preparing for the final exam in his history of France course. The exam will consist of 5 essay questions selected at random from a list of 10 the professor has handed out in advance. Not exactly a Napoleon buff, Peter has only prepared eight of the questions. Let \( y \) denote the number of questions on the exam which Peter has prepared.

\[ y \text{ is a hypergeometric r.v. with } N = 10, r = 8, n = 5. \]

1f. \( p(2) = \frac{4}{5}, p(-1) = \frac{3}{5} \) and hence

\[ E(y) = 2 \cdot \frac{4}{5} + (-1) \cdot \frac{3}{5} = -\frac{25}{5} = -0.4 \]

2. Businesses commonly project revenues under alternative economic scenarios. For a stylized example, inflation could be high or low and unemployment could be high or low. There are four possible scenarios, with the assumed probabilities:

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Inflation</th>
<th>Unemployment</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>high</td>
<td>high</td>
<td>0.20</td>
</tr>
<tr>
<td>2</td>
<td>high</td>
<td>low</td>
<td>0.20</td>
</tr>
<tr>
<td>3</td>
<td>low</td>
<td>high</td>
<td>0.36</td>
</tr>
<tr>
<td>4</td>
<td>low</td>
<td>low</td>
<td>0.24</td>
</tr>
</tbody>
</table>

Hint: Denote with \( A \) the event “high inflation”, and with \( B \) the event “high unemployment”.

2a [3pts] \( P(A) = P(A \cap B) + P(A \cap B^c) = 0.20 + 0.20 = 0.4. \)

2b [4pts] \( P(B) = P(A \cap B) + P(A^c \cap B) = 0.20 + 0.36 = 0.56 \) and hence \( P(A|B) = P(A \cap B)/P(B) = 0.20/0.56 = 0.36. \)
3c [3pts] Are inflation and unemployment independent? No, because \( P(A) = 0.4 \neq P(A|B) = 0.36 \).

3. A family has two dogs (Rex and Rover) and a little boy (Russ). None of them is fond of the mailman. Given that they are outside, Rex and Rover have a 30% and a 40% chance, respectively, of biting the mailman. Russ, if he is outside, has a 15% chance of doing the same thing. Suppose only one of the three is outside when the mailman comes. Rex is outside 50% of the time, Rover 20% of the time and Russ 30% of the time.

3a [5pts] What is the probability the mailman will be bitten? Denote with \( A_1 \), \( A_2 \), \( A_3 \) the events that Rex, Rover and Russ, respectively are outside. Denote with \( B \) the event that the mailman is bitten.

\[
P(B) = \sum_{i=1}^{3} P(B|A_i)P(A_i) = 0.3 \cdot 0.5 + 0.4 \cdot 0.2 + 0.15 \cdot 0.3 = 0.275
\]

3b [5pts] If the mailman is bitten, what are the chances that Russ did it?

\[
P(A_3|B) = \frac{P(B|A_3)P(A_3)}{\sum_{i=1}^{3} P(B|A_i)P(A_i)} = \frac{0.15 \cdot 0.3}{0.275} = 0.16
\]

4. James Bond insists that his martinis be shaken, not stirred. A skeptical bartender tests Bond with 6 martinis (using six coin flips to determine which drinks to shake and which to stir). Bond errs on one and correctly identifies the other 5 before passing out. Denote with \( p \) the probability that Bond can tell the difference between shaken and stirred Martinis.

4a [3pts] If \( p = 0.5 \), what is the probability of guessing 5 or more Martinis correctly?

Let \( y = \# \) correctly guesses. \( y \) is a binomial r.v. with \( n = 6 \), \( p = 0.5 \).
Using the tables in the appendix we find \( P(y \geq 5) = 1 - \sum_{k=0}^{4} p(k) = 0.11 \).
(or, use the Minitab command: \( \text{pdf; binomial n=6, p=0.5} \))

4b [3pts] Find the value of \( p \) such that guessing 5 out of 6 Martinis correctly is highest.

To find \( \hat{p} \) such that \( p(5) \) is maximized we consider \( p(5) \) as a function of \( p \) and maximize it in \( p \).
It is easier to maximize \( f(p) = \log p(5) \) instead of \( p(5) \):

\[
f(p) = \log \left[ c \cdot \hat{p}^5 (1-p)^1 \right] = \log(c) + 5 \log(p) + \log(1-p),
\]

where \( c \) is a factor which does not depend on \( p \) (and is hence irrelevant for the maximization).
To find the maximum of \( f(p) \), get the first derivative, and set it equal to zero:

\[
\frac{df}{dp} = \frac{5}{p} - \frac{1}{1-p} = 0 \quad \Rightarrow \quad p = \frac{5}{6}
\]

4c [2pts] Assume that instead of deciding initially to serve 6 Martinis, the bartender was serving Martinis until Bond guessed 5 correctly. Assuming \( p = 0.5 \), find the probability of erring once.

Denote with \( x \) the number of trials until Bond correctly guesses five times. \( x \) is a negative binomial r.v. with \( r = 5 \) and \( p = 0.5 \). Erring once implies \( x = 6 \), and thus:

\[
p_x(6) = 5 \cdot \hat{p}^5 (1-p)^1 = \frac{5}{2^6} = 5/64 = 0.08
\]

4d [2pts] Under the assumptions of 4c, find the value of \( p \) such that guessing 5 out of 6 Martinis correctly is highest.

The answer is the same as for 4b because \( p_x(6) = 1 \cdot \hat{p}^5 (1-p)^1 \) and \( p_y(5) = c \cdot \hat{p}^5 (1-p)^1 \), i.e., as a function of \( p \) the probability mass functions for \( x \) and \( y \) are proportional. Proportional functions have the same maximum.

\[\text{1Since the logarithm is a monotone transformation the value } p \text{ which maximizes the logarithm also maximizes the original function.}\]