These solutions were prepared by students studying to take the 2008 FYE. These solutions have not been reviewed by faculty members. Therefore, the solutions presented may contain errors. Use at your own risk.

1. \( p(y|\phi) \sim N(0, 1/\phi), \phi \sim Ga(1/2, 1/2), p(y) = \int p(y|\phi)p(\phi)d\phi \sim t(1, 0, 1) \)

b \( p(\phi|y) = \frac{p(y|\phi)p(\phi)}{p(y)} \propto \sqrt{\phi} \exp\{-\frac{\phi y^2}{2}\}(1+y^2)^{-1} = \exp\{-\frac{\phi(y^2+1)}{2}\}(1+y^2) \sim Exp(y^2+1) \)

c \( p(x|y, \phi) = \frac{p(x,y|\phi)}{p(y|\phi)} = \frac{1}{\sqrt{1-r^2}} \exp\{-\frac{\phi}{2(1-r^2)}(x^2 - 2rxy + y^2) + \phi y^2/2\} \sim N(ry, \frac{1-r^2}{\phi}) \)

d \( p(x|y) = \int p(x|y, \phi)p(\phi|y)d\phi \propto (1 + \frac{1}{2} \frac{(x-ry)^2}{1-r^2(y^2-n)})^{-3/2} \sim t_2(ry, \frac{(1-r^2)(y^2-n)}{2}) \)

e (I) \( \text{cov}(x, y) = E[E(x - Ex)(y - Ey)|\phi] = E[r\phi|\phi] = 0 \) They are uncorrelated with each other.

(II) \( x|\phi \sim N(0, 1/\phi), p(x) = \int p(x|\phi)p(\phi)d\phi = \frac{2}{1+r^2}(1+\frac{r^2}{1-r^2})^{-3/2}, \) while \( p(x|y) \sim t_2(ry, \frac{(1-r^2)(y^2-n)}{2}), p(x|y) \neq p(x), \) so they are not independent.
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2. (a)

\[
f(X_1, \ldots, X_{n+m} \mid \alpha_0, \alpha_1, \beta_0, \beta_1, \sigma^2) = \left( \frac{1}{2\pi\sigma^2} \right)^{n/2} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (Y_i - [\alpha_0 + \alpha_1 X_i])^2 \right\} \\
\times \left( \frac{1}{2\pi\sigma^2} \right)^{m/2} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=n+1}^{n+m} (Y_i - [\beta_0 + \beta_1 X_i])^2 \right\}.
\]

(b)

\[
\log f(X_1, \ldots, X_{n+m} \mid \alpha_0, \alpha_1, \beta_0, \beta_1, \sigma^2) = -\frac{n}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (Y_i - [\alpha_0 + \alpha_1 X_i])^2 + \\
-\frac{m}{2} \log(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=n+1}^{n+m} (Y_i - [\beta_0 + \beta_1 X_i])^2
\]

Denote: \(\log f(X_1, \ldots, X_{n+m} \mid \alpha_0, \alpha_1, \beta_0, \beta_1, \sigma^2)\) by \(\log L\).

\[
\frac{\partial \log L}{\partial \alpha_0} = \frac{1}{2\sigma^2} \sum_{i=1}^{n} (Y_i - [\alpha_0 + \alpha_1 X_i])^2 \\
\sum_{i=1}^{n} Y_i - n\alpha_0 - \alpha_1 \sum_{i=1}^{n} X_i^2 = 0 \\
\alpha_0 = \frac{1}{n} \sum_{i=1}^{n} Y_i - \frac{\alpha_1}{n} \sum_{i=1}^{n} X_i
\]

\[
\frac{\partial \log L}{\partial \alpha_1} = \frac{1}{2\sigma^2} \sum_{i=1}^{n} X_i (Y_i - [\alpha_0 + \alpha_1 X_i])^2 \\
\sum_{i=1}^{n} Y_i X_i - n\alpha_0 \sum_{i=1}^{n} X_i - \alpha_1 \sum_{i=1}^{n} X_i^2 = 0 \\
\alpha_1 = \frac{\sum_{i=1}^{n} Y_i X_i \sum_{i=1}^{n} X_i}{\left( \sum_{i=1}^{n} X_i^2 \right) - \frac{n\alpha_0}{\sum_{i=1}^{n} X_i} \sum_{i=1}^{n} X_i}
\]

\[
\hat{\alpha}_0 = \left[ \frac{1}{n} \sum_{i=1}^{n} Y_i - \frac{\sum_{i=1}^{n} Y_i X_i \sum_{i=1}^{n} X_i}{n \sum_{i=1}^{n} X_i^2} \right] \frac{\sum_{i=1}^{n} X_i^2}{\sum_{i=1}^{n} X_i^2 + (\sum_{i=1}^{n} X_i)^2}
\]

\[
\hat{\alpha}_1 = \frac{\sum_{i=1}^{n} Y_i X_i \sum_{i=1}^{n} X_i}{\left( \sum_{i=1}^{n} X_i^2 \right) - \frac{n\hat{\alpha}_0}{\sum_{i=1}^{n} X_i} \sum_{i=1}^{n} X_i}
\]

Similarly,

\[
\hat{\beta}_0 = \left[ \frac{1}{m} \sum_{i=n+1}^{n+m} Y_i - \frac{\sum_{i=n+1}^{n+m} Y_i X_i \sum_{i=n+1}^{n+m} X_i}{m \sum_{i=n+1}^{n+m} X_i^2} \right] \frac{\sum_{i=n+1}^{n+m} X_i^2}{\sum_{i=n+1}^{n+m} X_i^2 + (\sum_{i=n+1}^{n+m} X_i)^2}
\]

\[
\hat{\beta}_1 = \frac{\sum_{i=n+1}^{n+m} Y_i X_i \sum_{i=n+1}^{n+m} X_i}{\left( \sum_{i=n+1}^{n+m} X_i^2 \right) - \frac{m\hat{\beta}_0}{\sum_{i=n+1}^{n+m} X_i^2} \sum_{i=n+1}^{n+m} X_i}
\]
\[ \frac{\partial \log L}{\partial \sigma^2} = \frac{-n}{2} \frac{1}{2\pi \sigma^2} 2\pi + \frac{\sum_{i=1}^{n} (Y_i - (\alpha_0 + \alpha_1 X_i))^2}{2(\sigma^2)^2} + \frac{-m}{2} \frac{1}{2\pi \sigma^2} 2\pi + \frac{\sum_{i=n+1}^{n+m} (Y_i - (\beta_0 + \beta_1 X_i))^2}{2(\sigma^2)^2} = 0, \]

\[ \hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (Y_i - (\hat{\alpha}_0 + \hat{\alpha}_1 X_i))^2 + \sum_{i=n+1}^{n+m} (Y_i - (\hat{\beta}_0 + \hat{\beta}_1 X_i))^2}{n + m}. \]

(c) \[ \alpha_0 + \alpha_1 \gamma = \beta_0 + \beta_1 \gamma \]
\[ \alpha_0 - \beta_0 + \gamma (\alpha_1 - \beta_1) = 0 \]
\[ \hat{\gamma} = \frac{\hat{\alpha}_0 - \hat{\beta}_0}{\hat{\alpha}_1 - \hat{\beta}_1} \]

(d) We can rewrite \( \hat{\alpha}_0 + \hat{\alpha}_1 \gamma - \hat{\beta}_0 - \hat{\beta}_1 \gamma \) as
\[ \begin{pmatrix} 1 & \gamma \end{pmatrix} \begin{pmatrix} \hat{\alpha}_0 \\ \hat{\alpha}_1 \end{pmatrix} - \begin{pmatrix} 1 & \gamma \end{pmatrix} \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} \]

The MLE for \((\hat{\alpha}_0, \hat{\alpha}_1)\) is \((X^T X)^{-1}X^T Y\) where \(X\) is a \((n \times 2)\) matrix, and \(Y\) is a \(n\)-vector. The MLE for \((\hat{\beta}_0, \hat{\beta}_1)\) is \((X^T X)^{-1}X^T Y\) where \(X\) is a \((m \times 2)\) matrix, and \(Y\) is a \(m\)-vector. Note that we have a linear combination of Normal distributions, therefore, the desired distribution is Normal. From part (c) we know: \( \hat{\alpha}_0 + \hat{\alpha}_1 \gamma - \hat{\beta}_0 - \hat{\beta}_1 \gamma = 0 \). Using this result we conclude: \[ E \left[ (\hat{\alpha}_0 + \hat{\alpha}_1 \gamma - \hat{\beta}_0 - \hat{\beta}_1 \gamma) \right] = 0. \]

In order to determine the variance, consider the expressions indicated in the end of the problem.

\[ \text{Var} \left[ \begin{pmatrix} \hat{\alpha}_0 + \hat{\alpha}_1 \gamma - \hat{\beta}_0 - \hat{\beta}_1 \gamma \end{pmatrix} \right] = \]
\[ = \text{Var}(\hat{\alpha}_0) + \gamma \text{Var}(\hat{\alpha}_1) + 2\gamma \text{Cov}(\hat{\alpha}_0, \hat{\alpha}_1) + \text{Var}(\hat{\beta}_0) + \gamma \text{Var}(\hat{\beta}_1) + 2\gamma \text{Cov}(\hat{\beta}_0, \hat{\beta}_1) \]
\[ = \frac{\sigma^2 \sum_{i=1}^{n} X_i^2}{n S_{xx,(i=1,...,n)}} + \gamma^2 \frac{\sigma^2}{S_{xx,(i=1,...,n)}} - 2\gamma \frac{\sigma^2 X}{S_{xx,(i=1,...,n)}} + \]
\[ + \frac{\sigma^2 \sum_{i=n+1}^{n+m} X_i^2}{m S_{xx,(i=n+1,...,n+m)}} + \gamma^2 \frac{\sigma^2}{S_{xx,(i=n+1,...,n+m)}} - 2\gamma \frac{\sigma^2 X}{S_{xx,(i=n+1,...,n+m)}} \]

(e) \[ \frac{Y_i - (\alpha_0 - \alpha_1 X_i)}{\sigma} \sim N(0,1); \frac{|Y_i - (\alpha_0 - \alpha_1 X_i)|^2}{\sigma^2} \sim \chi_n^2; \sum_{i=1}^{n} \frac{|Y_i - (\alpha_0 - \alpha_1 X_i)|^2}{\sigma^2} \sim \chi_n^2 \]
and \[ \sum_{i=1}^{n} \frac{|Y_i - (\alpha_0 - \alpha_1 X_i)|^2}{\sigma^2} \sim \chi_n^2. \] Similarly, \[ \sum_{i=n+1}^{n+m} \frac{|Y_i - (\beta_0 - \beta_1 X_i)|^2}{\sigma^2} \sim \chi_{n-2}^2 \] and \[ B = \sum_{i=n+1}^{n+m} \frac{|Y_i - (\beta_0 - \beta_1 X_i)|^2}{\sigma^2}. \] Using independence, we can express: \[ \frac{A}{\sigma} + \frac{B}{\sigma} \sim \chi_{2n+m-4}^2. \]

Previously we have found: \((\hat{\alpha}_0 + \hat{\alpha}_1 \gamma - \hat{\beta}_0 - \hat{\beta}_1 \gamma) \sim N(0, \sigma^2 V^*)\) where \(V^*\) is the expression depending on \(X_1, \ldots, X_{n+m}\) shown below.

\[ V^* = \frac{\sum_{i=1}^{n} X_i^2}{n S_{xx,(i=1,...,n)}} + \gamma^2 \frac{1}{S_{xx,(i=1,...,n)}} - 2\gamma \frac{X}{S_{xx,(i=1,...,n)}} + \]
\[ + \frac{\sum_{i=n+1}^{n+m} X_i^2}{m S_{xx,(i=n+1,...,n+m)}} + \gamma^2 \frac{1}{S_{xx,(i=n+1,...,n+m)}} - 2\gamma \frac{X}{S_{xx,(i=n+1,...,n+m)}} \]
Note that:

$$\frac{\hat{\alpha}_0 + \hat{\alpha}_1 \gamma - \hat{\beta}_0 - \hat{\beta}_1 \gamma}{\sqrt{\sigma^2 V^*}} \sim N(0, 1);$$

$$\sqrt{\frac{\hat{\alpha}_0 + \hat{\alpha}_1 \gamma - \hat{\beta}_0 - \hat{\beta}_1 \gamma}{\sigma^2 V^*}} \frac{\sqrt{\frac{\lambda + \beta}{n + m - 4}}}{\sqrt{\frac{(A+B) V^*}{n+m-4}}} = \frac{\hat{\alpha}_0 + \hat{\alpha}_1 \gamma - \hat{\beta}_0 - \hat{\beta}_1 \gamma}{\sqrt{\frac{(A+B) V^*}{n+m-4}}} \sim t_{n+m-4}$$

The pivotal quantity is:

$$\frac{(\hat{\alpha}_0 + \hat{\alpha}_1 \gamma - \hat{\beta}_0 - \hat{\beta}_1 \gamma)^2}{\frac{(A+B) V^*}{n+m-4}} \sim F(1, n + m - 4)$$

(f) $V^*$ depends on $\gamma$. In order to build the confidence interval we will not be able to isolate $\gamma$ using the pivotal quantity above. Therefore, such interval does not exist.
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3. (a) Under $H_0$, the maximum likelihood for $\sigma$ is $\hat{\sigma} = \sigma_0$ due to the constraint that $\sigma = \sigma_0$. It is quite trivial to show that the maximum likelihood estimate for $\mu$ is $\bar{x}$.
Under $H_0$, the maximum likelihood estimate of $\hat{\mu} = \bar{x}$ and $\hat{\sigma} = \frac{\sum (x_i - \bar{x})^2}{n}$. These can be trivially found by taking the log of the likelihood, differentiating, setting equal to zero, and solving for $\hat{\mu}$ and $\hat{\sigma}$.

(b) The likelihood ratio statistic for testing $H_0$ versus $H_1$ is given by,

$$\Lambda = \frac{\sup_{H_0} L(\mu, \sigma)}{\sup_{H_1} L(\mu, \sigma)} \propto \frac{\sigma_0^{-n/2} \exp \left\{ -\frac{\sum (x_i - \bar{x})^2}{2\sigma_0^2} \right\}}{\hat{\sigma}^{-n/2} \exp \left\{ -\frac{\sum (x_i - \bar{x})^2}{2\hat{\sigma}^2} \right\}}$$

$$= \left( \frac{\sigma_0^2}{\hat{\sigma}^2} \right)^{-n/2} \exp \left\{ -\frac{\sum (x_i - \bar{x})^2}{2\sigma_0^2} + \frac{n}{2} \right\}$$

(c) Notice

$$\ln(\Lambda) = -\frac{n}{2} [\ln(\sigma_0^2) - \ln(\hat{\sigma}^2)] - \frac{\sum (x_i - \bar{x})^2}{2\sigma_0^2} + \frac{n}{2}$$

$$= c + \frac{n}{2} \ln(T(x)/n) - \frac{T(x)}{2\sigma_0^2}$$

Thus, the only function of the data in the log likelihood is $T(x)$. Thus, the rejection region for a size $\alpha$ test will base the rejection region off of $T(x)$.

(d) Notice $T(x) = x'(I_n - (1/n)J)x$ is a quadratic form where $I_n$ is the rank $n$ identity matrix and $J$ is an $n \times n$ matrix of ones. Therefore, according to the theory of quadratic forms,

$$x'(I_n - (1/n)J) \sim \chi^2_{n-1}$$

because $(I_n - (1/n)J)$ is idempotent of rank $n - 1$. Thus, critical values for this test based on $T(x)$ would be given from the $\chi^2$ distribution with $n - 1$ degrees of freedom.
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4. (a) 

\[
p(\mu | Y_S, \phi) \propto p(Y_S | \mu, \phi) p(\mu, \phi) \\
\propto \phi^{n/2} \exp \left\{ -\frac{\phi}{2} \left( -2 \mu \sum_{i \in S} Y_i + n \mu^2 \right) \right\} \frac{1}{\phi} \\
\propto \exp \left\{ -\frac{n \phi}{2} (\mu - \bar{Y}_S)^2 \right\} \\
\Rightarrow \mu | Y_S, \phi \sim N(\bar{Y}_S, \sigma^2/n)
\]

(b) 

\[
p(\phi | Y_S) \propto \int p(Y_S | \mu, \phi) p(\mu, \phi) d\mu \\
\propto \phi^{n/2-1} \exp \left\{ -\frac{\phi}{2} \left( \sum_{i \in S} Y_i^2 - n \bar{Y}_S^2 \right) \right\} \int_{-\infty}^{\infty} \exp \left\{ -\frac{n \phi}{2} (\mu - \bar{Y}_S)^2 \right\} d\mu \\
\propto \phi^{n/2-1} \exp \left\{ -\frac{\phi}{2} \left( n - 1 \right) s^2 \right\} (n\phi)^{-1/2} \\
\Rightarrow \phi | Y_S \sim \text{Ga}(n-1)/2, (n-1)s^2/2 \\
\Rightarrow \phi | Y_S \sim \frac{1}{(n-1)s^2} \chi^2_{n-1}
\]

(c) 

\[
p(Y_i | Y_S, \phi) \propto \int p(Y_i | \mu, \phi) p(\mu | Y_S, \phi) p(\phi | Y_S) d\mu \\
\propto \int_{-\infty}^{\infty} \exp \left\{ -\frac{\phi}{2} (Y_i - \mu)^2 \right\} \exp \left\{ -\frac{n \phi}{2} (\mu - \bar{Y}_S)^2 \right\} d\mu \\
\propto \exp \left\{ -\frac{\phi}{2} Y_i^2 \right\} \int_{-\infty}^{\infty} \exp \left\{ -\frac{\phi}{2} (-2 \mu Y_i + \mu^2 + n \mu^2 - 2n \mu \bar{Y}_S) \right\} d\mu \\
\propto \exp \left\{ -\frac{\phi}{2} \left( Y_i^2 - \frac{(n \bar{Y}_S + Y_i)^2}{n + 1} \right) \right\} \\
\times \int_{-\infty}^{\infty} \exp \left\{ -\frac{(n+1)\phi}{2} \left[ \mu - \frac{n \bar{Y}_S + Y_i}{n + 1} \right]^2 \right\} d\mu \\
\propto \exp \left\{ -\frac{\phi}{2} \left[ n Y_i^2 - 2n \bar{Y}_S Y_i \right] \right\} \\
\propto \exp \left\{ -\frac{n \phi}{2(n+1)} (Y_i - \bar{Y}_S)^2 \right\} \\
\Rightarrow Y_i | Y_S, \phi \sim N \left( \bar{Y}_S, \left( \frac{n+1}{n} \right) \sigma^2 \right)
\]

(d) Since \((\bar{Y} | Y_S, \phi)\) is a linear function of \((\bar{Y}_S, Y_S, \phi)\) (as shown in the hint), we only need to find the distribution of the latter random variable; we have
\[ \bar{Y}_{S_1} | \mathbf{Y}_S, \mu, \phi \sim N \left( \mu, \left( (N - n) \phi \right)^{-1} \right) \]

Thus

\[
p(\bar{Y}_{S_1} | \mathbf{Y}_S, \phi) \propto \int p(\bar{Y}_{S_1} | \mu, \phi)p(\mu | \mathbf{Y}_S, \phi)p(\phi | \mathbf{Y}_S)d\mu
\]

\[
\propto \int_{-\infty}^{\infty} \exp \left\{ -\frac{(N - n) \phi}{2} (\bar{Y}_{S_1} - \mu)^2 \right\} \exp \left\{ -\frac{n \phi}{2} (\mu - \bar{Y}_S)^2 \right\} d\mu
\]

\[
\propto \exp \left\{ -\frac{(N - n) \phi \bar{Y}_{S_1}^2}{2} \right\}
\]

\[
\propto \int_{-\infty}^{\infty} \exp \left\{ -\frac{\phi}{2} \left( -2(N - n) \mu \bar{Y}_{S_1} + (N - n) \mu^2 + n \mu^2 - 2n \mu \bar{Y}_S \right) \right\} d\mu
\]

\[
\propto \exp \left\{ -\frac{(N - n) \phi \bar{Y}_{S_1}^2}{2} + \phi (n \bar{Y}_S + (N - n) \bar{Y}_{S_1})^2 \right\}
\]

\[
\propto \exp \left\{ -\frac{\phi (N - n) \bar{Y}_{S_1}^2}{2} + \phi (N - n)^2 \bar{Y}_{S_1}^2 \right\}
\]

\[
\propto \exp \left\{ -\frac{\phi (N - n)}{2} \left[ 1 - \frac{N - n}{N} \right] \bar{Y}_{S_1}^2 + \phi \left[ 1 - \frac{n}{N} \right] \bar{Y}_S \bar{Y}_{S_1} \right\}
\]

\[
\propto \exp \left\{ -\frac{\phi (N - n)}{2} \left[ 1 - \frac{n}{N} \right] (\bar{Y}_{S_1} - \bar{Y}_S)^2 \right\}
\]

\[\Rightarrow \bar{Y}_{S_1} | \mathbf{Y}_S, \phi \sim N \left( \bar{Y}_S, \left( n \phi \left[ 1 - \frac{n}{N} \right] \right)^{-1} \right) \]

\[
\bar{Y} = \frac{n \bar{Y}_S + (N - n) \bar{Y}_{S_1}}{N}
\]

\[
\mathbb{E}[\bar{Y} | \mathbf{Y}_S, \phi] = \bar{Y}_S
\]

\[
\text{Var}[\bar{Y} | \mathbf{Y}_S, \phi] = \left[ \frac{N - n}{N} \right] \left[ \frac{N - n}{N} \right]^{-1} \frac{1}{n \phi}
\]

\[
= \left[ 1 - \frac{n}{N} \right] \frac{1}{n \phi}
\]

\[\Rightarrow \bar{Y} | \mathbf{Y}_S, \mu, \phi \sim N \left( \bar{Y}_S, \left[ 1 - \frac{n}{N} \right] \frac{\sigma^2}{n} \right) \]
\[ p(\bar{Y}|Y_S) \propto \int p(\bar{Y}|Y_S, \phi)p(\phi|Y_S)d\phi \]
\[ \propto \int_0^\infty \phi^{1/2} \exp \left\{ -\frac{n\phi}{2} \left[ \frac{N}{N-n} \right] (\bar{Y} - \bar{Y}_S)^2 \right\} d\phi \]
\[ \propto \phi^{(n-1)/2-1} \exp \left\{ -\frac{\phi(n-1)s^2}{2} \right\} d\phi \]
\[ \propto \int_0^\infty \phi^{n/2-1} \exp \left\{ -\frac{\phi}{2} \left[ \left( \frac{nN}{N-n} \right) (\bar{Y} - \bar{Y}_S)^2 + (n-1)s^2 \right] \right\} d\phi \]
\[ \propto \left[ \left( \frac{nN}{N-n} \right) (\bar{Y} - \bar{Y}_S)^2 + (n-1)s^2 \right]^{-(n-1)/2} \]
\[ \propto \left[ 1 + \frac{1}{n-1} \frac{n}{s^2} \left[ \frac{N}{N-n} \right] (\bar{Y} - \bar{Y}_S)^2 \right]^{-(n-1)/2} \]
\[ \Rightarrow \bar{Y}|Y_S \sim t\left( n-1, \bar{Y}_S, \left[ 1 - \frac{n}{N} \right] \frac{s^2}{n} \right) \]

Since the t-distribution becomes closer to the normal distribution as the number of degrees of freedom increases, for \( n \) large, we have that \( (\bar{Y}|Y_S) \) is approximately normal:

\[ \bar{Y}|Y_S \sim N\left( \bar{Y}_S, \left[ 1 - \frac{n}{N} \right] \frac{s^2}{n} \right) \]
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5. (a) The likelihood function is:

\[ f(U_1, \ldots, U_n \mid \theta) = \prod_{i=1}^{n} \frac{1}{\theta} 1(U_i \leq \theta) 1(U_i > 0) = \frac{1}{\theta^n} 1(U_{(n)} \leq \theta) 1(U_{(1)} > 0). \]

If \( U_{(n)} \leq \theta \), then \( f(U_1, \ldots, U_n \mid \theta) = \frac{1}{\theta^n} \) which is a decreasing function of \( \theta \). The largest value of \( f(U_1, \ldots, U_n \mid \theta) \) is associated to the smallest \( \theta \in [U_{(n)}, \infty) \).

If \( U_{(n)} > \theta \), then \( f(U_1, \ldots, U_n \mid \theta) = 0 \).

Conclusion: \( R_n(\vec{U}) = U_{(n)} = \max\{U_1, \ldots, U_n\} \) is the MLE.

(b) \( E(U) = \frac{\theta}{2} \). The method of moments estimator is the solution to the equation \( E(U) = \sum_{i=1}^{n} \frac{U_i}{n} \).

\[ \hat{\theta} = \frac{2}{n} \sum_{i=1}^{n} U_i = S_n(\vec{U}). \]

(c)

\begin{align*}
\pi(\theta > t) &= t^{-1} \text{ for } t \geq 1 \\
\pi(\theta \leq t) &= 1 - t^{-1} = F_\theta(t) \text{ for } t \geq 1 \\
\frac{d}{dt} F_\theta(t) &= \frac{t - (t - 1)}{t^2} = \frac{1}{t^2} \\
\pi(\theta) &= \frac{1}{\theta^2} \text{ for } \theta \geq 1
\end{align*}

\[ \pi(\theta \mid \vec{U}) = \frac{f(\vec{U} \mid \theta)}{\int_{1}^{\theta} f(\vec{U} \mid \theta) \pi(\theta) \, d\theta}. \]

\[ = \frac{1}{\theta^n} \times \frac{1}{\theta^2} = \frac{1}{\theta^n + 2}. \]

Last expression is the p.d.f. of Pareto(1, n + 1).

\[ E(\theta \mid \vec{U}) = \int_{1}^{\infty} \theta \frac{(n + 1)}{\theta^{n+2}} \, d\theta = \frac{n + 1}{n}. \]

(d) The p.d.f. of \( U_{(n)} \) is \( \frac{n}{\theta} \left( \frac{u}{\theta} \right)^{n-1} = n \frac{u^{n-1}}{\theta^n}, \text{ for } 0 \leq u \leq \theta. \)

\[ E(R_n(\vec{U})) = \int_{0}^{\theta} u \times n \frac{u^{n-1}}{\theta^n} \, du = \frac{n}{n + 1} \theta. \]

Therefore, \( R_n(\vec{U}) \) is biased.

\[ E(T_n(\vec{U})) = \frac{n + 1}{n} \times E(R_n(\vec{U})) = \frac{n + 1}{n} \times \frac{n}{n + 1} \theta = \theta. \]
\( T_n(\vec{U}) \) is Unbiased.

\[
E(S_n(\vec{U})) = \frac{2}{n} \sum_{i=1}^{n} E(U_i) = \frac{2}{n} \sum_{i=1}^{n} \frac{\theta}{2} = \theta.
\]

\( S_n(\vec{U}) \) is Unbiased.

(e) Chebychev’s Inequality

\[
P(|T(X) - E(T(X))| \geq \epsilon) \leq \frac{E|T(X) - E(T(X))|^2}{\epsilon^2} = \frac{E(T(X))^2 - E(T(X))^2}{\epsilon^2}.
\]

\[
E(R_n(\vec{U})^2) = \int_{0}^{\theta} u^2 \times n \frac{\theta^n - 1}{\theta^n} \, du = \frac{n}{n+2} \theta^2.
\]

In this case the Chebychev’s Inequality is given by

\[
P(|R_n(\vec{U}) - E[R_n(\vec{U})]| \geq \epsilon) \leq \frac{E[R_n(\vec{U})^2] - E[R_n(\vec{U})]^2}{\epsilon^2} = \frac{n}{n+2} \theta^2 - \frac{n^2}{(n+1)^2} \theta^2.
\]

\[
\lim_{n \to \infty} \frac{n}{n+2} \theta^2 - \frac{n^2}{(n+1)^2} \theta^2 = 0. \quad \text{As } n \to \infty, \quad P(|R_n(\vec{U}) - \frac{n\theta}{n+1}| \geq \epsilon) \to 0.
\]

For \( S_n(\vec{U}) \) we have

\[
\text{Var}(S_n(\vec{U})) = \frac{4}{n^2} \sum_{i=1}^{n} \text{Var}(U_i) = \frac{4}{n^2} n \frac{\theta^2}{12} = \frac{\theta^2}{3n}. \quad \lim_{n \to \infty} \frac{\theta^2}{3n} = 0. \quad \text{As } n \to \infty, \quad P(|S_n(\vec{U}) - \theta| \geq \epsilon) \to 0.
\]

For \( T_n(\vec{U}) \) we have

\[
\text{Var}(T_n(\vec{U})) = \left( \frac{n+1}{n} \right) \left[ \frac{n}{n+2} \theta^2 - \frac{n^2}{(n+1)^2} \theta^2 \right] = \frac{(n+1)^2}{n(n+2)} \theta^2 - \theta^2. \quad \lim_{n \to \infty} \frac{(n+1)^2}{n(n+2)} \theta^2 - \theta^2 = 0. \quad \text{As } n \to \infty, \quad P(|T_n(\vec{U}) - \theta| \geq \epsilon) \to 0.
These solutions were prepared by students studying to take the 2008 FYE. These solutions have not been reviewed by faculty members. Therefore, the solutions presented may contain errors. Use at your own risk.

6. Given that $X_1, \ldots, X_m \sim i.i.d. \text{Bin}(n, p)$ where both $n$ and $p$ are unknown.

(a) Likelihood:

$$L(n, p) = \prod_{i=1}^{m} \left( \binom{n}{X_i} p^{X_i} (1 - p)^{n - X_i} \right) = \prod_{i=1}^{m} \binom{n}{X_i} p^{\sum_{i=1}^{m} X_i} (1 - p)^{mn - \sum_{i=1}^{m} X_i}$$

(b) Given $n$, $[L(n, p) | n] \propto p^{\sum_{i=1}^{m} X_i} (1 - p)^{mn - \sum_{i=1}^{m} X_i}$.

Therefore the maximum likelihood estimate of $k$ given $n$ is the following:

$$\delta \frac{\partial [L(n, p) | n]}{\partial p} = 0 \Rightarrow \hat{p}(n) = \frac{\sum_{i=1}^{m} X_i}{mn}$$

(c) Profile likelihood

$$L^*(n) = L(n, \hat{p}(n)) = \prod_{i=1}^{m} \binom{n}{X_i} \left( \frac{\sum_{i=1}^{m} X_i}{mn} \right)^{\sum_{i=1}^{m} X_i} \left[ 1 - \frac{\sum_{i=1}^{m} X_i}{mn} \right]^{mn - \sum_{i=1}^{m} X_i}$$

(d) When $n$ is large and $p$ is small so that $\lambda = np$ remains fixed, we can use the Poisson approximation to the Binomial. The likelihood becomes:

$$L(n, p) \approx \prod_{i=1}^{m} e^{-np} \left( \frac{np}{X_i} X_i! \right) = e^{-mn p} \left( \frac{np}{\sum_{i=1}^{m} X_i} \right)^{\sum_{i=1}^{m} X_i} \prod_{i=1}^{m} X_i!$$

When $n$ is fixed we get MLE of $p$ by the following:

$$\delta \frac{\partial [L(n, p) | n]}{\partial p} = 0 \Rightarrow \hat{p}(n) = \frac{\sum_{i=1}^{m} X_i}{mn}$$

Note that this exactly the same with the exact likelihood result. The profile likelihood approximately becomes:

$$L^*(n) = L(n, \hat{p}(n)) \approx e^{-mn \frac{\sum_{i=1}^{m} X_i}{mn}} \left( \frac{\sum_{i=1}^{m} X_i}{mn} \right)^{\sum_{i=1}^{m} X_i} \prod_{i=1}^{m} X_i!$$

which is free of $n$. Therefore

$$L^*(\infty) = \lim_{n \to \infty} L^*(n) = e^{-\sum_{i=1}^{m} X_i} \left( \frac{\sum_{i=1}^{m} X_i}{m} \right)^{\sum_{i=1}^{m} X_i} \prod_{i=1}^{m} X_i!$$

(e) Given a fixed prior density $\pi_0$ for $(n, p)$ and $\epsilon > 0$ positive,

$$\Gamma = \{ \pi_k(n, p) = (1 - \epsilon) \pi_0(n, p) + \epsilon \delta_{k, \hat{p}(k)}(n, p) : k = 0, 1, 2, \ldots \}$$
Now
\[ \pi_k(n|X_1, \ldots, X_m) = \frac{\int_0^1 \pi_k(n, p)L(n, p)dp}{\sum_{n=0}^{\infty} \int_0^1 \pi_k(n, p)L(n, p)dp} \]
\[ = \frac{\int_0^1 \pi_k(n, p) \prod_{i=1}^{m} \left( \left( \frac{n}{X_i} \right) p^{X_i}(1-p)^{n-X_i} \right) dp}{\sum_{n=0}^{\infty} \int_0^1 \pi_k(n, p) \prod_{i=1}^{m} \left( \left( \frac{n}{X_i} \right) p^{X_i}(1-p)^{n-X_i} \right) dp} \]

We focus our attention on:
\[ \int_0^1 [(1-\epsilon)\pi_0(n, p) + \epsilon \delta_k, \hat{\pi}(n, p)] \prod_{i=1}^{m} \left( \left( \frac{n}{X_i} \right) p^{X_i}(1-p)^{n-X_i} \right) dp \]
\[ = (1-\epsilon) \int_0^1 \pi_0(n, p) \prod_{i=1}^{m} \left( \left( \frac{n}{X_i} \right) p^{X_i}(1-p)^{n-X_i} \right) dp + \epsilon \delta_k \prod_{i=1}^{m} \left( \left( \frac{k}{X_i} \right) \hat{\pi}^{X_i}(1-\hat{\pi})(n-X_i) \right) \]

Therefore:
\[ \pi_k(n|X_1, \ldots, X_m) \]
\[ = \frac{(1-\epsilon) \int_0^1 \pi_0(n, p) \prod_{i=1}^{m} \left( \left( \frac{n}{X_i} \right) p^{X_i}(1-p)^{n-X_i} \right) dp + \epsilon \delta_k \prod_{i=1}^{m} \left( \left( \frac{k}{X_i} \right) \hat{\pi}^{X_i}(1-\hat{\pi})(n-X_i) \right)}{(1-\epsilon) \sum_{n=0}^{\infty} \int_0^1 \pi_0(n, p) \prod_{i=1}^{m} \left( \left( \frac{n}{X_i} \right) p^{X_i}(1-p)^{n-X_i} \right) dp + \epsilon \prod_{i=1}^{m} \left( \left( \frac{k}{X_i} \right) \hat{\pi}^{X_i}(1-\hat{\pi})(n-X_i) \right)} \]

Since
\[ \pi_0(n|X_1, \ldots, X_m) = \frac{\int_0^1 \pi_0(n, p) \prod_{i=1}^{m} \left( \left( \frac{n}{X_i} \right) p^{X_i}(1-p)^{n-X_i} \right) dp}{\sum_{n=0}^{\infty} \int_0^1 \pi_0(n, p) \prod_{i=1}^{m} \left( \left( \frac{n}{X_i} \right) p^{X_i}(1-p)^{n-X_i} \right) dp} \]
then \( \pi_k(n|X_1, \ldots, X_m) \)
\[ = \frac{(1-\epsilon) \sum_{n=0}^{\infty} \int_0^1 \pi_0(n, p) \prod_{i=1}^{m} \left( \left( \frac{n}{X_i} \right) p^{X_i}(1-p)^{n-X_i} \right) dp + \epsilon \prod_{i=1}^{m} \left( \left( \frac{k}{X_i} \right) \hat{\pi}^{X_i}(1-\hat{\pi})(n-X_i) \right) \times \pi_0(n|X_1, \ldots, X_m)}{(1-\epsilon) \sum_{n=0}^{\infty} \int_0^1 \pi_0(n, p) \prod_{i=1}^{m} \left( \left( \frac{n}{X_i} \right) p^{X_i}(1-p)^{n-X_i} \right) dp + \epsilon \prod_{i=1}^{m} \left( \left( \frac{k}{X_i} \right) \hat{\pi}^{X_i}(1-\hat{\pi})(n-X_i) \right) \times \delta_k(n)} \]
\[ = (1-\alpha) \pi_0(n|X_1, \ldots, X_m) + \alpha \delta_k(n) \]

with
\[ \alpha = \frac{\epsilon \prod_{i=1}^{m} \left( \left( \frac{k}{X_i} \right) \hat{\pi}^{X_i}(1-\hat{\pi})(n-X_i) \right)}{(1-\epsilon) \sum_{n=0}^{\infty} \int_0^1 \pi_0(n, p) \prod_{i=1}^{m} \left( \left( \frac{n}{X_i} \right) p^{X_i}(1-p)^{n-X_i} \right) dp + \epsilon \prod_{i=1}^{m} \left( \left( \frac{k}{X_i} \right) \hat{\pi}^{X_i}(1-\hat{\pi})(n-X_i) \right)} \]
or
\[
\frac{1 - \alpha}{\epsilon} = \frac{(1 - \epsilon) \sum_{n=0}^{\infty} \int_0^1 \pi_0(n, p) \prod_{i=1}^{m} \left( \binom{n}{X_i} \right) p^{X_i} (1 - p)^{n - X_i} dp}{\prod_{i=1}^{m} \left( \binom{k}{X_i} \hat{p}(k) X_i (1 - \hat{p}(k))^{n - X_i} \right)}
\]

(f) Since \( \pi_k(n|X_1, \ldots, X_m) = (1 - \alpha) \pi_0(n|X_1, \ldots, X_m) + \alpha \delta_k(n) \), then
\[
E_{\pi_k}[n|X_1, \ldots, X_m] = (1 - \alpha) E_{\pi_0}[n|X_1, \ldots, X_m] + \alpha k
\]

Note that \( \alpha > 0 \) since the numerator and the denominator are both sum/integral of p.m.f. with prior having support on \( \{0, 1, \ldots\} \times [0, 1] \), and \( \epsilon \in (0, 1) \).

Hence
\[
\lim_{k \to \infty} E_{\pi_k}[n|X_1, \ldots, X_m] = \infty \Rightarrow \sup_{\pi \in \Gamma} E_{\pi}[n|X_1, \ldots, X_m] = \infty
\]