Analyzing Stochastic Diffusion Processes

Introduction

- Many interesting ecological diffusions
- Emerging diseases - avian flu, H1N1 flu
- Exotic organisms - invasive plants, gypsy moths
- Size and age distributions
- Transformation of landscape, deforestation, land use classifications, urban growth
The objective

- Our objective: forecast likely spread in space and time with associated uncertainty
- Nonlinear, nonhomogeneous in space and time
- Explanatory covariates
- Start with deterministic integro-differential equations or with partial differential equations
- How to add uncertainty?
cont.

- Theoretical models, “varying” around them
- Too much simplification required to obtain analytical solutions
- Discretization to fit models
- Do we care of the deterministic equation? Should we just work with the discrete time version we want? Dynamic spatial models?
Hierarchical modeling

The hierarchical paradigm:

\[ \text{data} | \text{process, parameters} | \text{process} | \text{parameters} | \text{parameters} \]

- A paradigm shift - designed experiments to observational studies; controlled experiments to integrated (big picture) investigation
- Prior information from: empirical studies, mechanistic knowledge, ecological theory, etc.
- Multiple information sources
- Conditional uncertainty in components (a natural way to specify models)
- Different resolutions in space and time
- Structured dependence in space and time
- Complex dependence structure through latent variables
Computation

- Fit within a Bayesian framework; enables full inference, exact inference
- Model fitting, associated computation is challenging
- High dimension, sparsity, dimension reduction
- MCMC/Gibbs sampler model fitting
- Model validation? model comparison?
Dynamics

- Continuous space, discrete time, i.e., $w_t(s)$
- Without loss of generality $t = (1, 2, ..., T)$
- Envision $w_t(s)$ as a \textit{dynamical} process
- In fact, simplify to first order Markov, i.e., for locations $s_1, s_2, ..., s_n$, let $w_t = (w_t(s_1), w_t(s_2), ..., w_t(s_n))^T$. Then

\[
[w_t | w_0, w_1, ... w_{t-1}] = [w_t | w_{t-1}]
\]

- For example, $w_t = H w_{t-1} + \eta_t$ where $\eta_t(s)$ incorporates spatial structure
- A vector AR(1) model and $H$ is called the \textit{propagator} matrix
- ??Specifying $H$??
Specifying $H$

- $H = I$ - not stationary (explosive), no interaction across space and time, not realistic for most dynamic processes of interest

- $H = \text{Diag}(h)$ where $\text{Diag}(h)$ has diagonal elements $0 < h_i < 1$ - Still no interactions

- Integro-difference equation (IDE) dynamics:

\[ w_t(s) = \int h(s, r; \phi) w_{t-1}(r) dr + \eta_t(s) \]

- $h$ is a “redistribution kernel” that determines the rate of diffusion and the advection
cont

- If require \( w > 0 \), work with

\[
\log w_t(s) = \log(\int h(s, r; \phi)w_{t-1}(r)dr) + \eta_t(s)
\]

- Alternatively,

\[
v_t(s) = \int h(s, r; \phi)v_{t-1}(r)dr
\]

and

\[
\log w_t(s) = \log v_t(s) + \eta_t(s)
\]

- Discretization to obtain \( H \)

- Forms for \( h(s, r; \phi); h(s, r; \phi(r)) \), \( h_t(s, r; \phi) \)?
Recall linear PDE, $\frac{dw(s,t)}{dt} = h(s)w(s,t)$

Finite differencing yields
$w(s, t + \Delta t) - w(s, t) = h(s)w(s, t)\Delta t$, i.e.,
$w(s, t + 1) \approx \tilde{h}(s)w(s, t)$. Same limitations as above.

Need more general PDE’s

PDE can motivate IDE, can clarify $H$

“forward” vs. “backward” perspective

IDE’s can be specified directly without using PDE’s,
e.g., $h(s, r)$ can be a sum of a survival/spread term + a
birth/replenishment term
Diffusion PDE’s

- Diffusion in one dimension - Fick’s Law: diffusive flux from high concentration to low is $-\delta \frac{\partial w(x,t)}{\partial x}$ with $\delta$, the diffusion coefficient. Location varying diffusion $\delta(x)$

- And, diffusion equation is $\frac{\partial w}{\partial t} = -\frac{\partial \text{flux}}{\partial x}$, i.e.,
  \[ \frac{\partial w(x,t)}{\partial t} = \frac{\partial}{\partial x} \left( \delta(x) \frac{\partial w(x,t)}{\partial x} \right) \]

- That is, the 1-dim diffusion equation is
  \[ \frac{\partial w(x,t)}{\partial t} = \delta'(x) \frac{\partial w(x,t)}{\partial x} + \delta(x) \frac{\partial^2 w(x,t)}{\partial x^2} \]

- In 2-dim, diffusive flux is $-\delta(x,y) \nabla w(x,y,t)$ ($\nabla w(x,y,t)$ is the concentration gradient at time $t$)

- The resulting diffusion PDE is
  \[ \frac{\partial w(x,y,t)}{\partial t} = \frac{\partial}{\partial x} \left( \delta(x,y) \frac{\partial w(x,y,t)}{\partial x} \right) + \frac{\partial}{\partial y} \left( \delta(x,y) \frac{\partial w(x,y,t)}{\partial y} \right) \]
Discretizing the diffusion equation

- Complete the differentiation of the diffusion equation
- Yields second order partial derivatives, $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial^2 w}{\partial y^2}$
- Introduce $\Delta t$, $\Delta x$, $\Delta y$
- Replace $\partial$’s with finite differences (first forward and second order centered) - careful detail, ugly expression!
- After the smoke clears, we obtain $w_{t+\Delta t} = Hw_t$
- Again, add $\eta_t$
- We are back to our earlier redistribution form
Add growth rate

- Previous dynamics simply redistribute existing population spatially over time
- In many situations, there is also growth of the population
- Population growth can be captured by a logistic differential equation

\[ \frac{\partial w(s, t)}{\partial t} = rw(s, t)(1 - w(s, t)/K) \]

- \( r \) is the growth rate, \( K \) is the carrying capacity
- Add growth to the diffusion PDE for \( \frac{\partial w(s,t)}{\partial t} \)
- \( r(s) \), \( K(s) \)
Eurasian collared dove data

- An example from Wikle et al., using data from the Breeding Bird Survey (BBS)
- Escaped to U.S. from Bahamas, introduced in Florida, expanding dramatically across North America
- 4000+ routes in the survey (some sampled more than once per year, others not sampled in a given year), length of route is \( \approx 40 \) kms, 50 stops per route, count birds by sight for 3 minutes item 18 years: 1986-2003
- Route is a “point”, response at a point is a count
- Aggregate to grid boxes
- \( Z_{it} \) is count in box \( i \) in year \( t \), \( n_{it} \) is number of visits to cell \( i \) in year \( t \).
- \( \lambda_{it} \) is intensity for box \( i \) in year \( t \)
Modeling specifics

- \( Z_{it} \sim \text{Po}(n_{it}\lambda_{it}) \)
- \( \log \lambda_{it} = w_{it} + \varepsilon_{it} \)
- \( \varepsilon_{it} \) are i.i.d. (pure error or micro-scale variation)
- The focus is on the \( w_{it} \). They tell the diffusion story, i.e.,
  \( w_{t} = H(\delta)w_{t-1} + \eta_{t} \)
- Model for \( \eta_{t} \)?
- \( w_{0} \sim N(0, 10I) \)
- \( \delta \) is the vector of local diffusion coefficients, one for each grid cell
- A dimension reduction for \( \delta \); many possibilities here - basis functions, EOF's, predictive processes
Figure 8.1. Location of BBS survey routes (+) and observed Eurasian Coloured Dove count for years 1985–2003. The radius of the circles are proportional to the observed count.
Figure 8.2. Sum of BBS Eurasian Collared-Dove counts over space for years 1986–2003.
Figure 8.3. Log of Eastern Collared-Dove BBS counts aggregated to a grid for years 1986–2010.
Figure 8.7. Posterior mean of $\delta$, the diffusion coefficients.
Figure 8.8. Posterior mean of log(\(\mu\)) for years 1986-2003.
Enriching the modeling

- Again, we focus on $w(s, t)$
- $w(s, t)$ can arise as a mean model for a geostatistical model or in a space-time GLM (as in Wikle) or as a cumulative intensity $\Lambda(s, t)$ for a space-time point pattern (which drives the cumulative diffusion)
- A general diffusion PDE (nonstochastic) looks like
  \[
  \frac{\partial w(s, t)}{\partial t} = a(w(s, t), z(s, t), \theta) \text{ where } z(s, t) \text{ are other potential variables (} z(s, t) = t \text{ for example)}
  \]
- $\theta(s)$?, $\theta(s, t)$?
- How to make the PDE stochastic?
- For the remainder, we use the logistic DE, i.e., a diffusion driven by a logistic growth model
- Discretization as proposed above
First a DE

Ignoring location $s$ for the moment, we have:

$$dw(t) = a(w(t), t, \theta)dt \quad \text{with} \quad w(0) = w_0$$

Simplest way to add stochasticity is to make $\theta$ random.

Instead:

$$dw(t) = a(w(t), t, \theta)dt + b(w(t), t, \theta)dZ(t)$$

where $Z(t)$ is Brownian motion over $R^1$ with $a$ and $b$ the “drift” and “volatility” respectively. Now a stochastic differential equation (SDE)

$\theta$ would still be random
Next: \( dw(t) = a(w(t), t, \theta(t))dt \) where (with \( Z(t) \) is variance 1 Brownian motion)

\[
d\theta(t) = g(\theta(t), t, \beta)dt + h(\theta(t), \sigma)dZ(t)
\]

This includes the previous example.

For the logistic equation:

\[
dw(t) = \theta(t)w(t) \left[ 1 - \frac{w(t)}{K} \right] dt
\]

If \( \theta(t) = \mu + \zeta(t) \) with \( d\zeta(t) = -a\zeta(t)dt + \sigma\zeta dZ(t) \)
equivalently

\[
d\theta(t) = -\alpha(\mu - \theta(t))dt + \sigma\zeta dZ(t),
\]
a self-reverting Ornstein-Uhlenbeck (OU) process and \( \theta(t) \) is a stationary GP with

\[
cov(\theta(t), \theta(t')) = (\sigma^2/\alpha)\exp(-\alpha|t - t'|).
\]
Add space

Now, we add space. First,

$$dw(s, t) = a(w(s, t), t, \theta(s))dt \text{ with } w(s, 0) = w_0(s),$$

a PDE. Randomness through $\theta(s)$, a process realization, so $\theta(s)$ provides the spatial dependence. Hence, a stochastic process of differential equations.

Next,

$$dw(s, t) = a(w(s, t), t, \theta(s))dt + b(w(s, t), t, \theta(s))dZ(s, t)$$

Modeling $Z(s, t)$? For a fixed finite set of spatial locations assume independent Brownian motion at each location.

Or a discrete space approximation to spatial Brownian motion employing a Gaussian process (GP) on $R^2$.
Next,

\[ dw(s, t) = a(w(s, t), t, \theta(s, t))dt \]

where say

\[ d\theta(s, t) = \gamma(\theta(s, t) - \theta(s))dt + bdZ(s, t) \]

Again, \( \theta(s) \) is process realization

Now, \( \theta(s, t) \) given through an infinite dimensional SDE

This version produces a covariance function that is separable in space and time
Important points

- A differential equation in time at every spatial location, i.e., parameters indexed by location

- The parameters vary spatially as realizations of a spatial process

- Instead, the differential equation is a stochastic differential equation (SDE), e.g., a spatial Ornstein-Uhlenbeck (O-U) process

- For the logistic PDE, the rate parameter in the differential equation is assumed to change over time. It can be modeled as a realization of a spatio-temporal process

- It can be modeled using a SDE, yielding an SDE embedded within the differential equation
Spatio-temporal modeling settings for the above?

The usual “geostatistics” setting with observations at locations and times:

\[ Y(s, t) = \Lambda(s, t) + \epsilon(s, t) \]

with \( \Lambda(s, t) \) modeled through a differential equation as above, i.e., process model is a stochastic PDE.

Space-time point pattern setting with data of the form \((s_i, t_i), i = 1, 2, \ldots, n\), i.e., random locations at random times. Now, we model the space-time intensity associated with the point pattern, again, say \( \Lambda(s, t) \), through a differential equation as above resulting in a space-time Cox process.
A Geostatistical Example

- The logistic PDE in space and time:

\[
\frac{\partial \Lambda(t,s)}{\partial t} = r(t,s) \Lambda(t,s) \left[ 1 - \frac{\Lambda(t,s)}{K(s)} \right]
\]

- Time discretized to intervals \( \Delta t \), indexed as \( t_j, j = 0, 1, 2, \ldots, J \). At location \( s_i \), data \( Y(t_j, s_i) \)

- Dynamic model: \( Y(t_j, s_i) = \Lambda(t_j, s_i) + \varepsilon(t_j, s_i) \)

- Using Euler’s approximation yields difference equation:
\[
\Delta \Lambda(t_j, s) = r(t_{j-1}, s) \Lambda(t_{j-1}, s) \left[ 1 - \frac{\Lambda(t_{j-1}, s)}{K(s)} \right] \Delta t,
\]
\[
\Lambda(t_j, s) \approx \Lambda(0, s) + \sum_{l=1}^{j} \Delta \Lambda(t_l, s)
\]

- No spatial flux, no \( \Delta x \), no \( \Delta y \), \( \Lambda_t = \text{Diag}(h) \Lambda_{t-1} \)
cont.

- Can not add scaled Brownian motion to the logistic PDE.
- Instead a time-varying growth rate at each location
- An O-U process for $r(t, s)$:

  $$\frac{\partial r(t, s)}{\partial t} = \alpha_r (\mu_r(s) - r(t, s)) + \frac{\partial B(t, s)}{\partial t}$$

- Model the initial $\Lambda(0, s)$ and $K(s)$ as log-Gaussian spatial processes with regression forms for the means
- Below we assume $K(s)$ known and set to 1, i.e., 100% is the capacity.
- Similar modeling for $\mu_r(s)$
A Simulation Example

- 10 × 10 study region
- 44 locations over 30 time periods
- 4 sites for holdout
- Matérn covariance function used for \( \Lambda_0(s) \), for \( r(s) \),
  \( (\nu = 3/2 \text{ used in the example}) \)
- E.g., the space time covariance function for \( r(t, s) \)
  becomes \( \rho(t_{j_1} - t_{j_2}, s_{i_1} - s_{i_2}) = \sigma_r^2 \exp(-\alpha_r |t_{j_1} - t_{j_2}|) (\phi_r |s_{i_1} - s_{i_2}|)^\nu \kappa(\phi_r |s_{i_1} - s_{i_2}|) \)
- weak priors on \( \sigma^2 \)'s, weak log normal prior on \( \alpha_r \),
  discrete prior on \( \phi \)'s
**Posterior Inference**

Table 1: Parameters and their posterior inference for the simulated example

<table>
<thead>
<tr>
<th>Model Parameters</th>
<th>True Value</th>
<th>Posterior Mean</th>
<th>95% Equal-tail Interval</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_\Lambda$</td>
<td>-4.2</td>
<td>-4.14</td>
<td>(-4.88, -3.33)</td>
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<tr>
<td>$\sigma_\Lambda$</td>
<td>1.0</td>
<td>0.91</td>
<td>(0.62, 1.46)</td>
</tr>
<tr>
<td>$\phi_\Lambda$</td>
<td>0.7</td>
<td>0.77</td>
<td>(0.50, 1.20)</td>
</tr>
<tr>
<td>$\sigma_\varepsilon$</td>
<td>0.05</td>
<td>0.049</td>
<td>(0.047, 0.052)</td>
</tr>
<tr>
<td>$\mu_r$</td>
<td>0.24</td>
<td>0.24</td>
<td>(0.22, 0.26)</td>
</tr>
<tr>
<td>$\sigma_r$</td>
<td>0.08</td>
<td>0.088</td>
<td>(0.077, 0.097)</td>
</tr>
<tr>
<td>$\phi_r$</td>
<td>0.7</td>
<td>0.78</td>
<td>(0.60, 1.10)</td>
</tr>
<tr>
<td>$\alpha_r$</td>
<td>0.6</td>
<td>0.64</td>
<td>(0.51, 0.98)</td>
</tr>
</tbody>
</table>
Figure 2: Observed space-time geostatistical data at 4 locations, actual (dashed line) and fitted mean growth curves (solid line), and 95% predictive intervals (dotted line) by our model (16) for the simulated data example.
Figure 3: Hold-out space-time geostatistical data at 4 locations, actual (dashed line) and predicted mean growth curves (solid line) and 95% predictive intervals (dotted line) by our model (16) for the simulated data example.
Figure 4: Hold-out space-time geostatistical data at 4 locations, actual (dashed line) and predicted mean growth curves (solid line) and 95% predictive intervals (dotted line) by the benchmark model (20) for the simulated data example.
Space-time point patterns

- Spatio-temporal Cox process models using SDEŠs
- Differential equation models for cumulative intensity
- We have 21 years of urban development data for Irving, TX but we just show a simulation example with five years
Urban Development Problem

Residential houses in Irving, TX

1951

1956

1962

1968
Spatio-temporal Cox Process

In a study region $D$ during a period of $[0, T]$, $N_T$ events:

Point pattern: $X_T = \left\{ x_{1,t_1}, \ldots, x_{N_T,t_{N_T}} \right\}$

where $x_{i,t_i} = (x^1_i, x^2_i, t_i)$

$X_T$ is a Poisson process with inhomogeneous intensity

$$\Omega(t, s), s \in D, t \in [0, T]$$

Specifying the intensity?

$$\Omega(t, s) = f(t, \theta_l(t, s); l = 1, \ldots, p)$$

$\theta_l(t, s), s \in D, l = 1, \ldots, p$ are processes for parameters of interest.
The cumulative intensity

Discretize the spatio-temporal Cox process in time:

Spatial point pattern: \( X_{[t_1,t_2]} \) during \( t \in [t_1, t_2) \)

\[ x_i = (x_i^1, x_i^2), \quad x_i \in X_{[t_1,t_2]} \]

The cumulative intensity for \( X_{[t_1,t_2]} \) is

\[
\int_{t_1}^{t_2} \Omega(t, s) \, dt = \int_{t_1}^{t_2} f(t, \theta(l(t, s)); l = 1, \ldots, p) \, dt
\]

We consider models for the cumulative intensity

\[
\Lambda(t, s) = \int_0^t \Omega(\tau, s) \, d\tau
\]
Comments

- So house locations and times over $(0, T] \times D$
- Need a $\Delta t$ and an area $A$ in order to observe a point pattern
- If $\Omega(t, s) \geq 0$ then $\Lambda(t, s)$ increases in $t$; we do not allow house removal
- Work with cumulative intensity $\Lambda(t, s)$ - easier to think about mechanistically. In fact, $\Lambda(t_2, s) - \Lambda(t_1, s)$ provides the intensity for the interval $(t_1, t_2]$.
- Dynamics in $\Lambda(t, s)$ provide dynamics for the discretized spatial point process
Illustrative growth models (each of which has an explicit solution)

- **Exponential growth**
  \[
  \frac{d\Lambda(t, s)}{dt} = r(s) \Lambda(t, s)
  \]

- **Gompertz growth**
  \[
  \frac{d\Lambda(t, s)}{dt} = r(s)e^{-\alpha(s)t}\Lambda(t, s)
  \]

- **Logistic growth**
  \[
  \frac{d\Lambda(t, s)}{dt} = r(s)\Lambda(t, s) \left[1 - \frac{\Lambda(t, s)}{K(s)}\right]
  \]

  - local growth rate
  - local carrying capacity
Logistic Population Growth

\[ \frac{d\Lambda(t,D)}{dt} = r(D)\Lambda(t,D) \left[ 1 - \frac{\Lambda(t,D)}{K(D)} \right] \]

population growth at time \( t \)

current population at time \( t \)
growth rate for region \( D \)
carrying capacity for region \( D \)

Model for the aggregate intensity.
Proper Scaling

Local growth model should scale with the global growth model:

\[
\lim_{|\delta_s| \to 0} \frac{\Lambda (t, \delta_s)}{|\delta_s|} = \Lambda (t, s); \quad \text{cumulate}
\]

\[
\lim_{|\delta_s| \to 0} \frac{K (\delta_s)}{|\delta_s|} = K (s); \quad \text{cumulate}
\]

\[
\lim_{|\delta_s| \to 0} r(\delta_s) = r (s). \quad \text{average}
\]

\[
\lim_{|\delta_s| \to 0} \frac{d\Lambda (t, \delta_s)}{|\delta_s|} dt = \lim_{|\delta_s| \to 0} r(\delta_s) \frac{\Lambda (t, \delta_s)}{|\delta_s|} \left[ 1 - \frac{\Lambda (t, \delta_s)}{K (\delta_s) / |\delta_s|} \right] \Rightarrow
\]

\[
\frac{d\Lambda (t, s)}{dt} = r(s)\Lambda (t, s) \left[ 1 - \frac{\Lambda (t, s)}{K (s)} \right]
\]
Process Models for the Parameters

r(s), K(s) and initial intensity

\[ \Lambda_0 (s) = \int_{-\infty}^{0} \Omega (\tau, s) \, d\tau \]

are parameter processes which are modeled on log scale as

\[ \log \Lambda_0 (s) = \mu_\Lambda (s; \beta_\Lambda) + \theta_\Lambda (s), \quad \theta_\Lambda (s) \sim GP (0, C_\Lambda (\phi_\Lambda)) \]

\[ \log r (s) = \mu_r (s; \beta_r) + \theta_r (s), \quad \theta_r (s) \sim GP (0, C_r (\phi_r)) \]

\[ \log K (s) = \mu_K (s; \beta_K) + \theta_K (s), \quad \theta_K (s) \sim GP (0, C_K (\phi_K)) \]

Hence, given \( r (s), K (s) \) and \( \Lambda_0 (s) \)

the growth curve is fixed. Also, the \( \mu \)'s are trend surfaces.
The problem

- We want the differential equations to be dependent at every location BUT
- We would not insist that the process exactly follows a logistic differential equation at every location
- We want to introduce some noise so convert to an SDE
- Again a time varying rate at each location
- Again $Z(t(s)) = \log r(t(s))$ with

$$dZ(t(s)) = \xi(Z(t(s)) - \log r(s)) + \sigma_Z dW(t, s)$$

- Induces a stationary process with a separable space-time covariance function
Discretizing Time (Euler Approximation)

Back to the original model, the intensity for the spatial point pattern in a time interval:

\[
\Delta \Lambda_j (s) = \int_{t_{j,1}}^{t_{j,2}} \Omega(\tau, s) \, d\tau \approx \Omega(t_{j,1}, s) \Delta t
\]

Difference equation model:

\[
\Delta \Lambda_j (s) = r(s) \Lambda_{j-1} (s) \left[ 1 - \frac{\Lambda_{j-1} (s)}{K(s)} \right] \Delta t
\]

\[
\Lambda_j (s) = \Lambda_0 (s) + \sum_{l=1}^{j-1} \Delta \Lambda_l (s)
\]

- explicit transition
- a recursion
Discrete-time Model

- Model parameters and latent processes:

\[ \theta_r(s), \theta_K(s) \text{ and } \theta_\Lambda(s) \]
\[ \beta_\Lambda, \beta_r, \beta_r \]
\[ \phi_\Lambda, \phi_r, \phi_K \]

- Likelihood

\[
\prod_{j=1}^{J} \left\{ \exp \left( - \int_D \Delta \Lambda_j(s) \, ds \right) \prod_{i=1}^{n_j} \Delta \Lambda_j(x_{ji}) \right\} \cdot \exp \left( - \int_D \Lambda_0(s) \, ds \right) \prod_{i=1}^{n_0} \Lambda_0(x_{0i})
\]
Discretizing Space

Divide region $D$ into $M$ cells. Rescaling and assuming homogeneous intensity in each cell. We obtain (with $r(m)$, $k(m)$ average growth rate and cumulative carrying capacity):

$$\frac{d\Lambda(t, m)}{dt} = r(m)\Lambda(t, m)\left[1 - \frac{\Lambda(t, m)}{K(m)}\right]$$

with induced transition

$$\Delta\Lambda_j(m) = r(m)\Lambda_{j-1}(m) \left[1 - \frac{\Lambda_{j-1}(m)}{K(m)}\right] \Delta t.$$ 

The joint likelihood (product Poisson):

$$\prod_{j=1}^{J} \left[ \exp \left( - \sum_{m=1}^{M} \Delta\Lambda_j(m) A(m) \right) \prod_{m=1}^{M} \Delta\Lambda_j(m)^{n_{jm}} \right] \cdot \exp \left( - \sum_{m=1}^{M} \Lambda_0(m) A(m) \right) \prod_{m=1}^{M} \Lambda_0(m)^{n_{0m}},$$
Simulated Data Analysis

Initially and successive 5 years
Simulated Data Analysis: Estimation

<table>
<thead>
<tr>
<th>Model Parameters</th>
<th>True Value</th>
<th>Posterior Median</th>
<th>95% Equal-tail Interval</th>
</tr>
</thead>
<tbody>
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<td>$\beta_0$</td>
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<td>2.998</td>
<td>(2.815, 3.211)</td>
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<tr>
<td>$\beta_1$</td>
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<td>0.897</td>
<td>(0.741, 1.091)</td>
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<td>$\mu_r$</td>
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<td>-2.991</td>
<td>(-3.135, -2.855)</td>
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<td>$\mu_K$</td>
<td>5.0</td>
<td>5.011</td>
<td>(4.844, 5.188)</td>
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<td>$\psi_\Lambda$</td>
<td>$2.0 \times 10^{-3}$</td>
<td>$2.37 \times 10^{-3}$</td>
<td>(1.62 $\times 10^{-3}$, 3.23 $\times 10^{-3}$)</td>
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<tr>
<td>$\psi_r$</td>
<td>$1.0 \times 10^{-3}$</td>
<td>$1.35 \times 10^{-3}$</td>
<td>(9.07 $\times 10^{-3}$, 1.95 $\times 10^{-3}$)</td>
</tr>
<tr>
<td>$\psi_K$</td>
<td>$1.0 \times 10^{-3}$</td>
<td>$7.46 \times 10^{-4}$</td>
<td>(7.91 $\times 10^{-5}$, 2.18 $\times 10^{-3}$)</td>
</tr>
<tr>
<td>$\xi_\Lambda$</td>
<td>0.2</td>
<td>0.204</td>
<td>(0.171, 0.251)</td>
</tr>
<tr>
<td>$\xi_r$</td>
<td>0.2</td>
<td>0.288</td>
<td>(0.21, 0.376)</td>
</tr>
<tr>
<td>$\xi_K$</td>
<td>0.6</td>
<td>0.241</td>
<td>(0.148, 0.505)</td>
</tr>
</tbody>
</table>
Simulated Data Analysis: Estimation

Posterior:

\[ \Lambda_0 \quad r \quad K \]

Actual:
Simulation: One Step Ahead Prediction

Predicted

Actual