GENERAL FREQUENTIST PROPERTIES OF THE
POSTERIOR PROFILE DISTRIBUTION

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In this paper, inference for the parametric component of a semiparametric model based on sampling from the posterior profile distribution is thoroughly investigated from the frequentist viewpoint. The higher order validity of the profile sampler obtained in Cheng and Kosorok (2006) is extended to semiparametric models in which the infinite dimensional nuisance parameter may not have a root-\(n\) convergence rate. This is a nontrivial extension because it requires a delicate analysis of the entropy of the semiparametric models involved. We find that the accuracy of inferences based on the profile sampler improve as the convergence rate of the nuisance parameter increases. Another extension we obtain is that the parametric component is permitted to be multivariate rather than only univariate. We also establish that an exact frequentist confidence interval obtained by inverting the profile log-likelihood ratio can be estimated with higher order accuracy by the credible set of the same type obtained from the posterior profile distribution. Our theory is verified for several specific examples.

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1. Introduction. Semiparametric models have the form \( P = \{ P_{\theta, \eta} : (\theta, \eta) \in \Theta \times \mathcal{H} \} \), where \( \Theta \subset \mathbb{R}^d \) and \( \mathcal{H} \) is an arbitrary subset that is typically infinite dimensional. In this paper, interest will focus on the parametric component \( \theta \), while the nonparametric component \( \eta \) will be considered a “nuisance parameter.” Inference for \( \theta \) will be based on semiparametric maximum likelihood estimation via the profile likelihood \( pl_n(\theta) = \sup_{\eta \in \mathcal{H}} lik_n(\theta, \eta) \), where \( lik_n(\theta, \eta) \) is the full likelihood given \( n \) observations.

If we define \( \hat{\eta}_\theta = \arg\max_{\eta \in \mathcal{H}} lik_n(\theta, \eta) \), the maximum likelihood estimator for the full likelihood can be expressed as \( (\hat{\theta}_n, \hat{\eta}_n) \), where \( \hat{\eta}_n = \hat{\eta}_{\hat{\theta}_n} \). We will assume throughout this paper that evaluation of \( pl_n(\theta) \) is computationally feasible. While this is not always the case, it is quite often a reasonable assumption in practice because of the availability of procedures such as the stationary point algorithm (as used in [11], for example) or the iterative convex minorant algorithm introduced in [7], to find \( \hat{\eta}_\theta \).

Many of the advantages of using the profile sampler for inference on \( \theta \) are discussed in [12]. The main argument is that direct maximization of the full likelihood and direct computation of the efficient Fisher information function, which often requires tedious evaluation of infinite dimensional operators that may not have a closed form, can both be avoided completely by using the profile sampler. This follows because the profile sampler yields a first order correct approximation to the maximum likelihood estimator \( \hat{\theta}_n \) and consistent estimation of the efficient Fisher information for \( \theta \), even when the nuisance parameter is not estimable at the \( \sqrt{n} \) rate.

The first order validity of this procedure established by [12] is extended to second order validity in [5] when the infinite dimensional nuisance param-
eter achieves the parametric rate. Specifically, higher order estimates of the maximum profile likelihood estimator and of the efficient Fisher information are obtained in \cite{5}. Moreover, \cite{5} also proves that an exact frequentist confidence interval for the parametric component at level $\alpha$ can be estimated by the $\alpha$ level credible set from the profile sampler with an error of order $O_P(n^{-1})$. Three rather different semiparametric models, the Cox model with right censored data, the proportional odds model with right censored data and case-control studies with a missing covariate, are studied in \cite{5}. Such higher order frequentist validity has not yet been established in semiparametric models for any other inferential approach, including the bootstrap. This presents a further argument in favor of using the profile sampler, at least when $\sqrt{n}$ consistent nuisance parameters are involved.

A natural question is whether the second order extension in \cite{5} can be further extended to settings where the nuisance parameter has arbitrary convergence rates, in particular, rates that are slower than the parametric rate. This extension is the key purpose of this paper. Additionally, we generalize the results to allow for multivariate parametric components (only univariate components were permitted in \cite{5}) and also show that another type of confidence interval for $\theta$, obtained by inverting the profile log-likelihood ratio, can also be estimated with higher order accuracy by the profile sampler.

In this paper, the convergence rate for the nuisance parameter $\eta$ is defined as the largest $r$ that satisfies $\|\hat{\eta}_n - \eta_0\| = O_P(\|\tilde{\theta}_n - \theta_0\| + n^{-r})$, where $\eta_0$ is the true value of $\eta$ and $\| \cdot \|$ is a norm with definition depending on context, i.e., for a Euclidean vector $u$, $\| u \|$ is the Euclidean norm, and for an element of the nuisance parameter space $\eta \in \mathcal{H}$, $\| \eta \|$ is some chosen norm on $\mathcal{H}$. In regular
semiparametric models, which we can define without loss of generality to be models where the entropy integral converges, \( r \) is always larger than \( 1/4 \). This lower bound for \( r \) is guaranteed by the continuity modulus of the empirical processes (see corollary 3.2.6 in [24]). The definition for the convergence rate of the nuisance parameter implicitly assumes that we have a norm defined on the set of possible values of the nuisance parameter \( \eta \). Obviously \( \eta_{\hat{\theta}_n} \overset{p}{\to} \eta_0 \) for any \( \hat{\theta}_n \overset{p}{\to} \theta_0 \). We say the nuisance parameter has parametric rate if \( r = 1/2 \). For instance, the nuisance parameters of the three examples in [5] achieve the parametric rate. More specifically, the nuisance parameter in the Cox model, which is the cumulative hazard function, has the parametric rate under right censored data. However, the convergence rate for the cumulative hazard becomes slower, i.e. \( r = 1/3 \), under current status data. The result is not surprising since current status data cannot provide as much information as right censored data.

Obviously our results for \( r = 1/2 \) coincide with the results in [5]. It is also no surprise that the accuracy of the profile sampler is dependent on the convergence rate of the nuisance parameter. The precise error rate for many of the quantities we study is \( O_P(M_n(r)) \), where we define \( M_n(r) = n^{-1/2} + n^{-2r+1/2} \) with support \( r > 1/4 \). Note that \( M_n(r) \) increases in \( r \) over the interval \( 1/4 < r < 1/2 \) and is constant for \( r \geq 1/2 \). Although we cannot yet prove it, we conjecture that this error rate is sharp, in the sense that when the error is multiplied by \( M_n^{-1}(r) \), it converges to a nondegenerate random quantity as \( n \to \infty \).

Perhaps the most important new result in this paper involves a comparison between an exact, frequentist confidence interval and a credible set for
\( \theta \) generated from the profile sampler. Specifically, we show that any rectangular credible set for \( \theta \) of level \( 1 - \alpha \) based on the profile sampler is within \( O_P(n^{-1/2} M_n(r)) \) of an exact, frequentist, rectangular confidence region with coverage \( 1 - \alpha \). Note that the choice of a one-sided credible set at a given level is not unique when the parameter dimension is \( \geq 2 \). We also establish higher order accuracy for the confidence interval obtained by inverting the profile log-likelihood ratio, defined as \( PLR_{\ell}(\theta) = 2(\log pl_n(\hat{\theta}_n) - \log pl_n(\theta)) \).

The next section, section 2, provides some necessary background material on semiparametric models, least favorable submodels, and empirical processes. The main concepts are illustrated with three examples which will be used throughout the paper. The primary assumptions required for the results of the paper are also be presented, along with a key tool for obtaining rates of convergence. In section 3, second order asymptotic expansions of the log-profile likelihood are presented. In section 4, we present the main result of the paper that the confidence interval for the parametric component of a semiparametric model can be approximated by the credible set based from the profile sampler with error of order \( O_P(M_n(r)) \). In section 5, we establish that the required assumptions are satisfied for the three previously introduced examples. Section 6 contains a brief discussion of future research directions, and proofs are given in section 7.

2. Background and Assumptions. We assume the data \( X_1, \ldots, X_n \) are i.i.d. throughout the paper. In what follows, we first briefly review the concept of a least favorable submodel. We then present three different examples for which we discuss the forms of the least favorable submodel and related model specifications. Next, we present the model assumptions needed.
for the remainder of the paper, and, finally, we give a key tool for the rate of convergence calculations needed in later sections.

2.1. The Least Favorable Submodel. A submodel $t \mapsto p_{t,\eta_t}$ is defined to be "least favorable" at $(\theta, \eta)$ if $	ilde{\ell}_{\theta, \eta} = \partial / \partial t \log p_{t,\eta_t}$, given $t = \theta$, where $	ilde{\ell}_{\theta, \eta}$ is the efficient score function for $\theta$. The efficient score function for $\theta$ can be viewed as the projection of the usual score function for $\theta$ onto the tangent space of $\eta$. The inverse of the variance of $	ilde{\ell}_{\theta, \eta}$ is exactly the efficient information matrix $\tilde{I}_{\theta, \eta}$. We also abbreviate $\tilde{\ell}_{\theta, \eta_t}$ and $\tilde{I}_{\theta, \eta_t}$ with $\tilde{\ell}_0$ and $\tilde{I}_0$, respectively. The "direction" along which $\eta_t$ approaches $\eta$ in the least favorable submodel is called "the least favorable direction." An insightful review of least favorable submodels and efficient score functions can be found in chapter 3 of [9]. A systematic coverage of semiparametric efficiency theory can be found in [2] and [3].

The least favorable submodel in this paper will be constructed in the following manner. We first assume the existence of a smooth map from the neighborhood of $\theta$ into the parameter set for $\eta$, of the form $t \mapsto \eta_t(\theta, \eta)$, such that the map $t \mapsto \ell(t, \theta, \eta)(x)$ can be defined as follows:

$$\ell(t, \theta, \eta)(x) = \log \text{lik}(t, \eta_t(\theta, \eta))(x),$$

where $t$ and $\theta$ are allowed to be multi-dimensional in this paper, although they both must have the same dimension, and where we require $\eta_0(\theta, \eta) = \eta$ for all $(\theta, \eta) \in \Theta \times \mathcal{H}$. We will now illustrate the form of this map for several examples, and the remaining requirements for the map will be presented when the model assumptions are listed later on in this section.
2.2. Examples. Three examples with different convergence rates are presented in this subsection. The Cox model with right censored data, which has a parametric convergence rate, has previously been studied in [5]. Nevertheless, it will be useful to review this example briefly here, although most of the details are given in [5]. The second example, the Cox model with current status data, has a cube-root convergence rate for the nuisance parameter. The last example is the partly linear regression model with normal residual error, where the convergence rate of the nuisance parameter is $n^{-2/5}$ under current status data.

2.2.1. Example 1. The Cox model with right censored data. In the Cox proportional hazards model, the hazard function of the survival time $T$ of a subject with covariate $Z$ is expressed as:

$$\lambda(t|z) = \lim_{\Delta \to 0} \frac{1}{\Delta} \Pr(t \leq T < t + \Delta | T \geq t, Z = z) = \lambda(t) \exp(\theta z),$$

where $\lambda$ is an unspecified baseline hazard function and $\theta$ is a vector including the regression parameters [6]. For the Cox model applied to right-censored failure time data, we observe $X = (Y, \delta, Z)$, where $Y = T \wedge C, \delta = I\{T \leq C\}$, and $Z \in Z \subset \mathbb{R}^d$ is a regression covariate. The cumulative hazard function $\Lambda(y) = \int_0^y \lambda(t) dt$ is considered the nuisance parameter, and the maximum likelihood estimator of $\Lambda$, $\hat{\Lambda}_n$, is a nondecreasing step function with support points at the observed event times. The convergence rate of the estimated nuisance parameter is established in [8]:

$$(2) \quad \|\hat{\Lambda}_{\theta_n} - \Lambda_0\|_\infty = O_P(n^{-\frac{1}{2}} + \|\hat{\theta}_n - \theta_0\|).$$

Expression (2) directly implies consistency of $\hat{\Lambda}_n$ in the uniform norm.
Based on the model assumptions specified in section 5.1 of [5], we can express the likelihood for \((\theta, \eta)\) in the following form:

\[
lik(\theta, \Lambda) = \left( e^{\theta_z \Lambda(y)} e^{-e^{\theta_z \Lambda(y)}} \right)^{\delta} \left( e^{-e^{\theta_z \Lambda(y)}} \right)^{1-\delta},
\]

by dropping the factors involving the distribution of \((C, Z)\) and replacing \(\lambda(y)\) by the point mass \(\Lambda\{y\}\). Hence the score functions for \(\theta\) and \(\Lambda\) can be easily derived as:

\[
\hat{\nu}_{\theta, \Lambda}(x) = \delta z - e^{\theta_z \Lambda(y)},
\]

\[
A_{\theta, \Lambda} h(y, \delta, z) = \delta h(y) - e^{\theta_z} \int_{[0, y]} h d\Lambda.
\]

Again, by the derivations in section 5.1 of [5], the least favorable direction at \((\theta, \Lambda)\), denoted \(h_{\theta, \Lambda}\), can be shown to be

\[
h_{\theta, \Lambda}(y) = \frac{E_{\theta, \Lambda} e^{\theta Z} Z 1\{Y \geq y\}}{E_{\theta, \Lambda} e^{\theta Z} 1\{Y \geq y\}}.
\]

If we let \(h_0\) denote the least favorable direction at the true parameters, the least favorable submodel \(\ell(t, \theta, \Lambda)\) has the form

\[
\ell(t, \theta, \Lambda) = \log lik(t, A_t(\theta, \Lambda)),
\]

where \(t \mapsto A_t(\theta, \Lambda) = \Lambda + (\theta - t) h_0\). Note that we have tacitly swapped the notation for \(\eta\) with \(\Lambda\) since \(\Lambda\) is more widely used in this context.

2.2.2. *Example 2. The Cox model with current status data.* Current status data arises when each subject is observed at a single examination time, \(Y\), to determine if an event has occurred. The event time, \(T\), cannot be known exactly. If a vector of covariates, \(Z\), is also available, then the observed data are \(n\) independent and identically distributed realizations of
$X = (Y, \delta, Z) \in \mathbb{R}^+ \times \{0, 1\} \times \mathbb{R}$, where $\delta = I\{T \leq Y\}$. The model of the conditional hazard given $Z$ is the same as in the previous example. Throughout the remainder of the discussion, we make the following assumptions. $T$ and $Y$ are independent given $Z$. $Z$ lies in a compact set almost surely and the covariance of $Z - E(Z|Y)$ is positive definite, which guarantees the efficient information $\tilde{I}_0$ to be positive definite. $Y$ possesses a Lebesgue density which is continuous and positive on its support $[\sigma, \tau]$, for which the true nuisance parameter $\Lambda_0$ satisfies $\Lambda_0(\sigma-) > 0$ and $\Lambda_0(\tau) < M < \infty$, and this density is continuously differentiable on $[\sigma, \tau]$ with derivative bounded above and bounded below by zero. Under these assumptions the maximum likelihood estimator of $(\theta, \Lambda)$ exists, $\hat{\theta}_n$ is asymptotically efficient and $\|\hat{\Lambda}_n - \Lambda_0\|_{L_2} = O_P(n^{-\frac{1}{2}})$, where $\| \cdot \|_{L_2}$ is the norm on $L_2([\sigma, \tau])$. Note that the conditions on the density of $Y$ ensure that $\|\Lambda - \Lambda_0\|_{L_2}$ is equivalent to $(\int_\sigma^\tau (\Lambda(y) - \Lambda_0(y))^2 dF^Y(y))^{1/2}$, where $F^Y(y)$ is the distribution of the observation time $Y$. Moreover, using entropy methods, [15] who extend earlier results of [8], show that

$$\|\hat{\Lambda}_{\theta_n} - \Lambda_0\|_{L_2} = O_P(\|\hat{\theta}_n - \theta_0\| + n^{-1/3}).$$

It is not difficult to derive that the log-likelihood for $n$ i.i.d. samples is

$$\log l_{ikn}(\theta, \Lambda) =$$

$$\sum_{i=1}^{n} \delta_i \log [1 - \exp(-\Lambda(Y_i) \exp(\theta Z_i))] - (1 - \delta_i) \exp(\theta Z_i) \Lambda(Y_i).$$

The score function takes the form $\dot{L}_{\theta, \Lambda}(x) = z \Lambda(y) Q(x; \theta, \Lambda)$, where

$$Q(x; \theta, \Lambda) = e^{\theta z} \left[ \delta \frac{\exp(-\theta z \Lambda(y))}{1 - \exp(-\theta z \Lambda(y))} - (1 - \delta) \right].$$

Inserting a submodel $t \mapsto \Lambda_t$ such that $h(y) = -\partial/\partial t|_{t=0} \Lambda_t(y)$ exists for every $y$, into the log likelihood, and differentiating at $t = 0$, we obtain a
score function for $\Lambda$ of the form $A_{\theta, \Lambda} h(x) = h(y)Q(x; \theta, \Lambda)$. The linear span of these functions contains $A_{\theta, \Lambda} h$ for all bounded function $h$ of bounded variation. The efficient score function for $\theta$ is defined as $\ell_{\theta, \Lambda} = \hat{\ell}_{\theta, \Lambda} - A_{\theta, \Lambda} h_{\theta, \Lambda}$ for the vector of functions $h_{\theta, \Lambda}$ minimizing the distance $P_{\theta, \Lambda} \| \hat{\ell}_{\theta, \Lambda} - A_{\theta, \Lambda} h \|^2$, which is also called the least favorable direction. The solution at the true parameter $(\theta_0, \Lambda_0)$ is $h_0(Y)$ defined as follows:

$$y \mapsto h_0(y) = \Lambda_0(y) h_{00}(y) = \Lambda_0(y) \frac{E_{\theta_0, \Lambda_0}(Z Q^2(X; \theta_0, \Lambda_0) | Y = y)}{E_{\theta_0, \Lambda_0}(Q^2(X; \theta_0, \Lambda_0) | Y = y)}.$$ 

As the formula shows, the vector of functions $h_0(y)$ is unique a.s., and $h_0(y)$ is a bounded function since $Q(x; \theta_0, \Lambda_0)$ is bounded away from zero and infinity. We shall assume the function $y \mapsto h_0(y)$ given by (9) has a version which is differentiable with a bounded derivative on $[\sigma, \tau]$.

The least favorable submodel can be defined as

$$\ell(t, \theta, \Lambda) = \log lik(t, \Lambda_t(\theta, \Lambda)),$$

where $\Lambda_t(\theta, \Lambda) = \Lambda + (\theta - t)\phi(\Lambda) \left( h_{00} \circ \Lambda_0^{-1} \right) \circ \Lambda$, and $\phi(\cdot)$ is a specially constructed function that smoothly approximates the identity. The function $\Lambda_t(\theta, \Lambda)$ is essentially $\Lambda$ plus a perturbation in the least favorable direction, $h_0$, but its definition is somewhat complicated in order to ensure that $\Lambda_t(\theta, \Lambda)$ really defines a cumulative hazard function within our parameter space, at least for $t$ that is sufficiently close to $\theta$. The details on the construction of the least favorable submodel can be found on page 23 of [14].

2.2.3. Example 3. Partly linear normal model with current status data.

In this example, a continuous outcome $Y$, conditional on the covariates $(W, Z) \in \mathbb{R}^d \times \mathbb{R}$, is modeled as:

$$Y = \theta^T W + k(Z) + \xi,$$
where \( k \) is an unknown smooth function, and \( \xi \sim N(0,1) \). Note that the choice \( N(0,1) \) is needed for model identifiability. We are interested in the regression parameter \( \theta \) and consider \( k(\cdot) \) to be an infinite dimensional nuisance parameter. This model is essentially a generalized partly linear model with normal link function. However, the response \( Y \) is not observed directly, but only its current status is observed at a random censoring time \( C \in \mathbb{R} \). In other words, we observe \( X = (C, \Delta, W, Z) \), where \( \Delta = 1_{Y < C} \). Additionally \( (Y, C) \) is assumed to be independent given \( (W, Z) \). Although it is not difficult to generalize to multivariate \( \theta \), we restrict our attention to univariate \( \theta \) in what follows for ease of exposition.

Under the model (10), and after noting that the joint distribution for \( (C, W, Z) \) does not involve the parameters \((\theta, k)\), the log-likelihood for a single observation at \( X = x \equiv (c, \delta, w, z) \) can be shown to have the form

\[
\text{loglik}_{\theta, k}(x) = \delta \log \{ \Phi(c - \theta w - k(z)) \} \\
+ (1 - \delta) \log \{ 1 - \Phi(c - \theta w - k(z)) \},
\]

where \( \Phi \) is the standard normal distribution. We further assume that the joint distribution for \( (C, W, Z) \) is strictly positive and finite. The covariates \((W, Z)\) are assumed to belong to some compact set \( \mathcal{W} \times Z \subset \mathbb{R}^2 \). And the random censoring time \( C \) is assumed to have support \([l_c, u_c]\), where \(-\infty < l_c < u_c < \infty\). In addition, we assume \( E[\text{Var}(W|Z)] \) is strictly positive and \( Ek(Z) = 0 \).

The regression parameter \( \theta \) is assumed to belong to some compact set in \( \mathbb{R}^d \), and the functional nuisance parameter \( k \) is assumed to belong to \( O^2_2 \equiv \{ f : J_2(f) + \| f \|_{\infty} < M \} \) for a known \( M < \infty \). The \( m \)-th order
Sobolev norm of a function $f$, $J_m(f)$, is defined as:

$$J_m(f) = \left[ \int_Z (f^{(m)}(z))^2 dz \right]^{1/2}.$$ 

Here, $m$ is a fixed integer and $f^{(j)}$ is the $j$-th derivative of $f(\cdot)$ with respect to $z$. The $m$-th order Sobolev class of functions is the class of functions $f$ supported on some compact set on the real line with $J_m(f) < \infty$. Hence the class $\mathcal{O}_2^M$ is trivially the subset of a second order Sobolev class of functions, and $k \in \mathcal{O}_2^M$ has known upper bound for both its uniform norm and its Sobolev norm. Note that the asymptotic behavior of penalized log-likelihood estimates in this model have been extensively studied in [13].

We now introduce the least favorable submodel. The score function for $\theta$ is $\ell_{\theta,k} \equiv w Q(x; \theta, k)$, where

$$Q(x; \theta, k) = (1 - \Delta) \frac{\phi(q_{\theta,k}(X))}{1 - \Phi(q_{\theta,k}(X))} - \Delta \frac{\phi(q_{\theta,k}(X))}{\Phi(q_{\theta,k}(X))}$$

and $q_{\theta,k}(X) = C - \theta W - k(Z)$. Furthermore, by defining $k_t = k + th$ for $h \in \mathcal{O}_2^M$, we can obtain the score function for $k$ in the direction $h: A_{\theta,k}h(x) = h(z)Q(x; \theta, k)$. The least favorable direction $h_{\theta,k}$ minimizes $h \mapsto P_{\theta,k} \| \ell_{\theta,k} - A_{\theta,k}h \|^2$. By solving the equation $P_{\theta,k}(\ell_{\theta,k} - A_{\theta,k}h)A_{\theta,k}h = 0$, we can obtain the solution at the true parameter values:

$$h_0(z) = \frac{E_0(WQ^2(X; \theta, k)|Z = z)}{E_0(Q^2(X; \theta, k)|Z = z)},$$

where $E_0$ is the expectation relative to the true parameters. Thus the least favorable submodel can be constructed as

$$\ell(t, \theta, k) = \log \text{lik}(t, k_t(\theta, k)),$$

where $k_t(\theta, k) = k + (\theta - t)h_0$. 

(12)
Note that the above model would be more flexible if we did not require knowledge of \( M \). A sieved estimator could be obtained if we replaced \( M \) with a sequence \( M_n \to \infty \). The theory we propose in this paper will be applicable in this setting, but, in order to maintain clarity of exposition, we have elected not to pursue this more complicated situation here. Another alternative approach is to use penalization. However, this is beyond the scope of the present paper.

2.3. Assumptions. We now present the assumptions that will be used throughout the paper, along with some necessary notation. The dependence on \( x \in \mathcal{X} \) of the likelihood and score quantities will be largely suppressed for clarity in this section and hereafter.

For the vector \( V \), matrix \( M \) and tensor \( T \), the notation \( V_i, M_{i,j} \) and \( T_{i,j,k} \) indicate its \( i \)-th, \((i,j)\)-th and \((i,j,k)\)-th element, respectively. \( M^T \) represents the transpose of the matrix \( M \). If each random variable in the matrix is of the same order \( O_P(\tau_n) \), we say that matrix is also of that order. The derivative of the log-likelihood of the least favorable submodel is with respect to the first argument, \( t \). The quantities \( \ell(t, \theta, \eta) \), \( \ell'(t, \theta, \eta) \) and \( \ell''(t, \theta, \eta) \) are separately the first, second and third derivative of \( \ell(t, \theta, \eta) \) with respect to \( t \). For brevity, we denote \( \ell_0 = \ell(\theta_0, \theta_0, \eta_0) \), \( \ell_0' = \ell'(\theta_0, \theta_0, \eta_0) \) and \( \ell_0'' = \ell''(\theta_0, \theta_0, \eta_0) \), where \( \theta_0, \eta_0 \) are the true values of \( \theta \) and \( \eta \). Of course, \( \ell_0(X) \) can also be written as \( \ell_0(X) \) based on the construction of the least favorable submodel. Obviously, \( \ell(t, \theta, \eta) \) and \( \ell'(t, \theta, \eta) \) are separately a \( d \)-dimensional vector and square matrix of \( d \times d \) dimension. The quantity \( \ell''(t, \theta, \eta) \) is a tensor. We thus define \( V^T \otimes P\ell''(t, \theta, \eta) \otimes V \) as a \( d \)-dimensional vector whose \( i \)-th element equals \( V^T(\partial^2/\partial t^2)(P\ell(t, \theta, \eta))_i V \). Similarly \( V^T \otimes P\ell''(t, \theta, \eta) \)
is a square matrix whose $(i,j)$-th element is $V^T(\partial / \partial t_j)(\partial^2 / \partial t_i \partial t_j) \ell(t, \theta, \eta)$. We use $\ell_{t_{i}, t_{j}, t_{k}}(t, \theta, \eta)$ to denote $(\partial^3 / \partial t_i \partial t_j \partial t_k) \ell(t, \theta, \eta)$. For the derivatives relative to the other two arguments, $\theta$ and $\eta$, we use the following shortened notation: $\ell_{\theta}(t, \theta, \eta)$ indicates the first derivative of $\ell(t, \theta, \eta)$ with respect to $\theta$. Similarly, $\ell_{t_{i}, \theta}(t, \theta, \eta)$ denotes the derivative of $\ell(t, \theta, \eta)$ with respect to $\theta$. Also, $\ell_{t_{i}}(\theta)$ and $\ell_{t_{j}}(\eta)$ indicate the maps $\theta \mapsto \tilde{\ell}(t, \theta, \eta)$ and $\eta \mapsto \ell_{t_{i}}(t, \theta, \eta)$, respectively. Let the random vector $\theta_n$ denote $\bar{\theta}_0^{1/2}(\theta - \hat{\theta}_n)$, and let $\phi_d(\cdot)$ ($\Phi_d(\cdot)$) represent the density (cumulative distribution) of a $d$-dimensional standard normal random variable ($N_d(0, I)$). The notations $\gtrsim$ and $\lesssim$ mean $\geq$, or $\leq$, up to a universal constant. Define $x \lor y$ ($x \land y$) to be the maximum (minimum) value of $x$ and $y$. The symbols $\bar{P}_n$ and $C_n \equiv \sqrt{n}(\bar{P}_n - P)$ are used for the empirical distribution and the empirical process of the observations, respectively.

We now make the following assumptions in order to achieve the desired second order asymptotic expansions of the log-profile likelihood (22). The assumption A2 below guarantees that the least favorable submodel passes through $(\theta, \eta)$:

**Regular Assumptions:**

A1 : $\theta_0 \in \Theta \subset \mathbb{R}^d$, where $\Theta$ is a compact and $\theta_0$ is an interior point of $\Theta$.

A2 : $\eta_\theta(\theta, \eta) = \eta$ for any $(\theta, \eta) \in \Theta \times \mathcal{H}$.

A3 : $\tilde{I}_0$ is positive definite.

We next describe the smoothness conditions for the least favorable submodel. Clearly, the assumptions B1 and B2 below are separately the smoothness conditions for the Euclidean parameter $(t, \theta)$ and the infinite dimensional nuisance parameter $\eta$. In principle, these assumptions directly imply
the no-bias conditions:

\[
\mathbb{P}_n \hat{\ell}(\theta_0, \hat{\theta}_n, \eta_{\hat{\theta}_n}) = \mathbb{P}_n \tilde{\ell}_0 + O_P(n^{-1/2} + n^{-r} + \| \hat{\theta}_n - \hat{\theta}_n \|^2),
\]

\[
\mathbb{P}_n \ddot{\ell}(\theta_0, \hat{\theta}_n, \eta_{\hat{\theta}_n}) = P \ddot{\ell}_0 + O_P(n^{-1/2} + n^{-r} + \| \hat{\theta}_n - \hat{\theta}_n \|^2),
\]

for \( \hat{\theta}_n \xrightarrow{p} \theta_0 \), thus making the profile likelihood behave like a standard parametric likelihood asymptotically.

**Smoothness Assumptions:**

B1: The maps

\[
(t, \theta, \eta) \mapsto \frac{\partial^{l+m}}{\partial t^l \partial \theta^m} \ell(t, \theta, \eta)
\]

have integrable envelope functions in \( L_1(P) \) in some neighborhood of \((\theta_0, \theta_0, \eta_0)\), for \((l, m) = (0, 0), (1, 0), (2, 0), (3, 0), (1, 1), (1, 2), (2, 1)\).

B2: Assume:

\[
\mathbb{G}_n(\ell(\theta_0, \theta_0, \hat{\theta}_n) - \hat{\ell}_0) = O_P(M_n(r) + (n^{1/2 - r} \vee 1) \| \hat{\theta}_n - \theta_0 \|),
\]

\[
P \ddot{\ell}(\theta_0, \theta_0, \eta) - P \ddot{\ell}(\theta_0, \theta_0, \eta_0) = O(\| \eta - \eta_0 \|),
\]

\[
P \dot{\ell}(\theta_0, \theta_0, \eta) - P \dot{\ell}(\theta_0, \theta_0, \eta_0) = O(\| \eta - \eta_0 \|^2),
\]

for \( \hat{\theta}_n \xrightarrow{p} \theta_0 \) and all \( \eta \) in some neighborhood of \( \eta_0 \).

There are three approaches to verifying the smoothness assumption (13), which is essential a continuity modulus of \( |\mathbb{G}_n \hat{\ell}(\theta_0, \theta_0, \eta) - \mathbb{G}_n \hat{\ell}_0| \). If the nuisance parameter has parametric convergence rate, we only need to show that the class of functions

\[
\left\{ \frac{\ell(\theta_0, \theta_0, \eta) - \hat{\ell}_0}{\| \eta - \eta_0 \|} : \text{for } \eta \text{ in some neighborhood of } \eta_0 \right\}
\]
belongs to a \( P \)-Donsker class. Alternatively, if the nuisance parameter has the cubic rate, the continuity modulus of the empirical process turns out to be of the order \( O_P(n^{-1/6} + n^{1/6} \| \tilde{\theta}_n - \theta_0 \|) \), or equivalently \( O_P(n^{-1/6} + \| \eta_{\tilde{\theta}_n} - \eta_0 \|^1) \). The method used to check this condition depends on the norm of the nuisance parameter and the bracketing entropy number of the class of functions \( G = \{ \ell(\theta_0, \theta_0, \eta) \text{ for } \eta \text{ in some neighborhood of } \eta_0 \} \). When \( \| \cdot \| \) is the \( L_2 \) norm or one of its dominating norms, we can make use of lemma 5.13 in [23]. Another approach is to calculate the order of \( E_Po \| G_n \|_F \), where \( F = \{(\ell(\theta_0, \theta_0, \eta_{\tilde{\theta}_n}) - \ell_0)/(M_n(r) + (n^{1/2-r} + 1)\|\tilde{\theta}_n - \theta_0\|)\} \), by the use of lemma 3.4.2 in [24]. The above three methods will be respectively employed later on in verifying the assumptions for the three main examples.

Boundedness of the Fréchet derivatives of the maps \( \eta \mapsto \ell(\theta_0, \theta_0, \eta) \) and \( \eta \mapsto \ell_t(\theta_0, \theta_0, \eta) \) is sufficient to ensure validity of conditions (14) and (15). Note that \( \ell_t(\theta_0, \theta_0, \eta_0) = 0 \) by the following analysis. Fixing \( \eta \) and differentiating \( P_{\theta, \eta} \ell(\theta, \theta, \eta) \) relative to \( \theta \) yields \( P_{\theta, \eta} \ell_{\theta, \eta}(\theta, \theta, \eta)T + P_{\theta, \eta} \ell(\theta, \theta, \eta) + \ell_{\theta}(\theta, t, \eta) = 0 \), since \( P_{\theta, \eta} \ell(\theta, \theta, \eta) = 0 \) for every \((\theta, \eta)\), and since we can choose \((\theta, \eta) = (\theta_0, \eta_0)\). One way to verify (16) is to write

\[
P \ell(\theta_0, \theta_0, \eta) = P \left[ \frac{P_0 - P_{\theta_0, \eta}}{P_0} (\ell(\theta_0, \theta_0, \eta) - \ell(\theta_0, \theta_0, \eta_0)) \right]
- P \left[ \ell(\theta_0, \theta_0, \eta_0) \left( \frac{P_{\theta_0, \eta} - P_0}{P_0} - A_0(\eta - \eta_0) \right) \right],
\]

where \( A_0 = A_{\theta_0, \eta_0} \) and \( A_{\theta, \eta} \) is the score operator for \( \eta \) at \((\theta, \eta)\), e.g., the Fréchet derivative of \( \log p_{\theta, \eta} \) relative to \( \eta \). Thus, if the \( L_2 \)-norm or one of its dominating norms is applied to \( \eta \), it suffices to show, under the given regularity conditions, Fréchet differentiability of \( \eta \mapsto \ell(\theta_0, \theta_0, \eta) \) plus second order Fréchet differentiability of \( \eta \mapsto lik(\theta_0, \eta) \).
Finally we assume that the following empirical processes conditions hold for \((t, \theta, \eta)\) in some neighborhood of their true values:

\textit{Empirical Processes Assumptions:}

C1 : There exists some neighborhood \(V\) of \((\theta_0, \theta_0, \eta_0)\) in \(\Theta \times \Theta \times \mathcal{H}\) such that the classes of functions \(\{\ell_1(t, \theta, \eta)(x) : (t, \theta, \eta) \in V\}\) and
\(\{(\ell_t)(t, \theta, \eta))_{i,j}(x) : (t, \theta, \eta) \in V\}\) are P-Donsker and
\(\{(\ell^{(3)}(t, \theta, \eta))_{i,j,k}(x) : (t, \theta, \eta) \in V\}\)

is P-Glivenko-Cantelli, for every \(i, j, k = 1, \ldots, d\).

One basic method of showing that a class of functions is P-Donsker or P-Glivenko-Cantelli involves calculating its (bracketing) entropy number. However this verification can be simplified by building up Glivenko-Cantelli (Donsker) classes from other Glivenko-Cantelli (Donsker) classes by employing preservation techniques in section 9.3 and 9.4 of [9]. In addition, the class \(BV_M\) of all functions \(f : [0, \tau] \mapsto \mathbb{R}\) that are uniformly bounded by a constant \(M\) and are of variation bounded by \(M\) is P-Donsker. Also, every P-Donsker class \(\mathcal{F}\) with integrable envelope function is P-Glivenko-Cantelli.

2.4. \textit{Rates of Convergence.} The estimation accuracy of the profile sampler method depends mainly on the convergence rate of the estimated nuisance parameter, i.e., the value of \(r\). We now present two useful results, theorem 1 and lemma 1 below, that are useful for determining this rate. These results are theorem 3.2 and lemma 3.3 in [15], and the proofs can be found therein. Theorem 1 is an extension from general results on M-estimators to semiparametric M-estimators with nuisance parameters:
THEOREM 1. Assume for any given \( \theta \in \Theta_n \) that \( \eta_\theta \) satisfies \( \mathbb{P}_n m_{\theta, \eta_0} \geq \mathbb{P}_n m_{\theta, \eta_0} \) for given measurable functions \( x \mapsto m_{\theta, \eta}(x) \). Assume conditions (17) and (18) below hold for every \( \theta \in \Theta_n \), every \( \eta \in \mathcal{V}_n \) and every \( \epsilon > 0 \):

\[ P(m_{\theta, \eta} - m_{\theta, \eta_0}) \preceq -d_0^2(\eta, \eta_0) + \|\theta - \theta_0\|^2, \]

\[ \sup_{\theta \in \Theta_n, \eta \in \mathcal{V}_n, \|\theta - \theta_0\| < \epsilon} |G_n(m_{\theta, \eta} - m_{\theta, \eta_0})| \preceq \phi_n(\epsilon). \]

Suppose that (18) is valid for functions \( \phi_n \) such that \( \delta \mapsto \phi_n(\delta)/\delta^\alpha \) is decreasing for some \( \alpha < 2 \) and sets \( \Theta_n \times \mathcal{V}_n \) such that \( P(\tilde{\theta} \in \Theta_n, \tilde{\eta}_\theta \in \mathcal{V}_n) \to 1 \). Then \( d_0(\tilde{\eta}_\theta, \eta_0) \leq O_P(\delta_n + \|\tilde{\theta} - \theta_0\|) \) for any sequence of positive numbers \( \delta_n \) such that \( \phi_n(\delta_n) \leq \sqrt{n} \delta^2_n \) for every \( n \).

Lemma 1 below is useful for verifying the continuity modulus condition (18) for the empirical process. Define \( \mathcal{S}_\delta = \{x \mapsto m_{\theta, \eta}(x) - m_{\theta, \eta_0}(x) : d_0(\eta, \eta_0) < \delta, \|\theta - \theta_0\| < \delta\} \) and

\[ K(\delta, \mathcal{S}_\delta, L_2(P)) = \int_0^\delta \sqrt{1 + H_B(\epsilon, \mathcal{S}_\delta, L_2(P))} d\epsilon. \]

**Lemma 1.** Suppose that the functions \( (x, \theta, \eta) \mapsto m_{\theta, \eta}(x) \) are uniformly bounded for \((\theta, \eta)\) ranging over a neighborhood of \((\theta_0, \eta_0)\) and that

\[ P(m_{\theta, \eta} - m_{\theta, \eta_0})^2 \preceq d_0^2(\eta, \eta_0) + \|\theta - \theta_0\|^2. \]

Then condition (18) is satisfied for any functions \( \phi_n \) such that

\[ \phi_n(\delta) \geq K(\delta, \mathcal{S}_\delta, L_2(P)) \left(1 + \frac{K(\delta, \mathcal{S}_\delta, L_2(P))}{\delta^2 \sqrt{n}}\right) \]

Consequently, we may replace \( \phi_n(\delta) \) with \( K(\delta, \mathcal{S}_\delta, L_2(P)) \) in the conclusion of the previous theorem.
3. Second Order Asymptotic Expansion. In this section, second order asymptotic expansions of the log-profile likelihood and maximum likelihood estimator are derived. Their second order accuracy is proven to be dependent on the order of the convergence rate of the nuisance parameter through the rate function $M_n(r)$ given in the introduction. Note that the smallest order of $O_P(M_n(r))$, $O_P(n^{-1/2})$, is achieved when the nuisance parameter has parametric or faster rate by the truncation property of the function $M_n(r)$. The assumptions in section 2 are assumed throughout.

**Theorem 2.** If $\tilde{\theta}_n$ satisfies $(\tilde{\theta}_n - \hat{\theta}_n) = o_P(1)$, then

\begin{align}
(21) \quad \sqrt{n}(\hat{\theta}_n - \theta_0) &= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{I}_0^{-1} \tilde{e}_0(X_i) + O_P(M_n(r)), \\
(22) \quad \log pl_n(\tilde{\theta}_n) &= \log pl_n(\hat{\theta}_n) - \frac{1}{2} n(\tilde{\theta}_n - \hat{\theta}_n)^T \tilde{I}_0(\hat{\theta}_n - \hat{\theta}_n)
+ O_P(g_r(\|\tilde{\theta}_n - \hat{\theta}_n\|)),
\end{align}

where $g_r(w) \equiv (nw^3 + n^{1-r}w^2 + n^{1-2r}w + n^{-2r+1/2})1\{1/4 < r < 1/2\} + (nw^3 + n^{-1/2})1\{r \geq 1/2\}$.

**Remark 1.** Under regularity conditions, the counterpart of (21) in fully parametric models has error of order $O_P(n^{-1/2})$, which agrees with $O_P(M_n(r))$ when $r \geq 1/2$. Thus, we achieve the parametric bound in semi-parametric models only when the nuisance parameter obtains the parametric rate. We also observe a monotonic increase in the error rate as $r$ decreases toward $1/4$.

The asymptotic quadratic expansion (22) can be used to construct an estimator of the standard error of $\hat{\theta}_n$. The estimator is the following “discretized” version of the observed profile information matrix, $\hat{I}_n$, which is the
derivative of the profile likelihood (see [15]):

\begin{equation}
\hat{I}_n(v) = -2 \frac{\log p l_n(\hat{\theta}_n + s_n v) - \log p l_n(\hat{\theta}_n)}{ns_n^2},
\end{equation}

where direction \( v \in \mathbb{R}^d \) and step size \( s_n \to 0 \). The expansion (22) implies

\begin{equation}
v^T \hat{I}_0 v = \hat{I}_n(v) + O_P(h_r(|s_n|)),
\end{equation}

where \( h_r(|s_n|) = g_r(|s_n|)/ns_n^2 \). By straightforward analysis, the smallest order of the error term in (24) is \( O_P(n^{-r}) \) by setting the step size \( s_n = O_p(n^{-r}) \) and \( s_n^{-1} = O_P(n^r) \) when \( 1/4 < r < 1/2 \). However, when \( r \geq 1/2 \), the smallest order of error in (24) stabilizes at \( O_P(n^{-1/2}) \) by setting the step size to \( s_n = O_P(n^{-1/2}) \) and \( s_n^{-1} = O_P(n^{1/2}) \). In other words, \( \hat{I}_n \) can only be a \( \sqrt{n} \) consistent estimator of \( \hat{I}_0 \) when the convergence rate of the nuisance parameter is faster than or equal to the parametric rate.

The above analysis also leads to good discretized estimators for each element in \( \hat{I}_0 \). For instance, with \( e_i \) denoting the \( i \)th unit vector in \( \mathbb{R}^d \), we can deduce

\begin{equation}
(\hat{I}_n(e))_{i,j} = - \frac{\log p l_n(\hat{\theta}_n + e_i s_n + e_j s_n) - \log p l_n(\hat{\theta}_n)}{ns_n^2} \hspace{1cm} + \frac{\log p l_n(\hat{\theta}_n + e_i s_n) + \log p l_n(\hat{\theta}_n + e_j s_n)}{ns_n^2},
\end{equation}

\begin{equation}
(\hat{I}_0)_{i,j} = (\hat{I}_n(e))_{i,j} + O_P(h_r(|s_n|)).
\end{equation}

4. Main results and implications. We now present the main results on the posterior profile distribution. Let \( P_{\theta|X} \) be the posterior profile distribution of \( \theta \) with respect to the prior \( \rho(\theta) \) given the data \( X = (X_1, \ldots, X_n) \). Define \( \Delta_n(\theta) = n^{-1} \{ \log p l_n(\theta) - \log p l_n(\hat{\theta}_n) \} \). We now present the first main result:

\[ \text{Imprint}: \text{imprint-os ver. 2006/01/04 file: paper2.tex date: December 6, 2006} \]
THEOREM 3. Assume the assumptions of section 2 and also that
\[ \Delta_n(\hat{\theta}_n) = o_P(1) \] implies that \( \hat{\theta}_n = \theta_0 + o_P(1), \)
for any sequence \( \hat{\theta}_n \in \Theta. \) If \( \rho(\theta_0) > 0 \) and \( \rho(\cdot) \) has continuous and finite first order derivative in some neighborhood of \( \theta_0, \) then
\[ \sup_{\xi \in \mathbb{R}^d} \left| \hat{F}_{\theta|X}(\sqrt{n}\hat{\theta}_n - \theta_0) - \Phi_d(\xi) \right| = O_P(M_n(r)). \]

REMARK 2. Based on the conclusions of theorem 3, we know that the \( [\alpha+O_P(M_n(r))] \)-th one-sided and two sided credible sets for vector \( \theta \) from the profile sampler are \( (-\infty, \hat{\theta}_n + n^{-1/2}\hat{\tau}^{-1/2}z_{\alpha}] \) and \( \{\hat{\theta}_n - n^{-1/2}\hat{\tau}^{-1/2}z_{(1-\alpha)/2}, \hat{\theta}_n + n^{-1/2}\hat{\tau}^{-1/2}z_{(1+\alpha)/2}\}, \) respectively, where \( z_{\alpha} \) is a standard normal \( \alpha \)-th quantile for \( d \)-dimensions and \( \hat{\tau} \) can be either \( \hat{\tau}_0 \) or \( \hat{\tau}_n. \)

The following two corollaries provide several interesting additional second order properties of the profile sampler:

COROLLARY 1. Assume the conditions of theorem 3, and let \( f_n(\cdot) \) be the posterior profile density of \( \sqrt{n}\hat{\theta}_n \) relative to the prior \( \rho(\theta). \) Then
\[ f_n(\xi) = \phi_d(\xi) + o_P(M_n(r)). \]

COROLLARY 2. Under the conditions of theorem 3 and recalling that \( \hat{\epsilon}_n = \hat{\tau}_0^{1/2}(\theta - \hat{\theta}_n), \) we have that if \( \theta \) has finite second absolute moment, then
\[ \hat{\theta}_n = \hat{E}_{\theta|X}(\theta) + o_P(n^{-1/2}M_n(r)), \]
\[ \hat{\tau}_0 = n^{-1}(\hat{\text{Var}}_{\theta|X}(\theta))^{-1} + o_P(M_n(r)), \]
where \( \hat{E}_{\theta|X}(\theta) \) and \( \hat{\text{Var}}_{\theta|X}(\theta) \) are the posterior profile mean and posterior profile covariance matrix, respectively.
Remark 3. The posterior moments in corollary 2 are with respect to the posterior profile distribution. Thus we can estimate \( \hat{\theta}_n \) with the mean of the profile sampler. Similarly, (each element of) the efficient information matrix can be estimated by (the corresponding element of) the inverse of the covariance matrix of the profile sampler with an error of order \( O_P(M_n(r)) \).

Clearly, a faster convergence rate of the nuisance parameter leads to higher estimation accuracy. We can generalize the arguments used in the proof of corollary 2 to obtain general results on the posterior moments. For simplicity, assume \( \theta \) is one dimensional. Then, provided \( \int_{-\infty}^{+\infty} |\theta|^{\alpha} \rho(\theta) d\theta < \infty \), we have

\[
E_{\theta|X} v_n^\beta = n^{-\beta/2} EU^\beta + O_P(n^{-(\beta+1)/2} + n^{(1-2\alpha-\beta)/2}),
\]

where \( E_{\theta|X} v_n^\beta \) is the \( \beta \)-th posterior moment of \( v_n \) and \( U \sim N(0,1) \).

Remark 4. We now have two approaches to estimating the efficient information matrix \( \tilde{I}_0 \). One approach is by numerical analysis as given in (26). Another approach is presented in corollary 2 as an estimate from the posterior distribution. We prefer estimating \( \tilde{I}_0 \) with (31) using the profile sampler procedure in semiparametric models with \( r \geq 1/2 \) since this avoids the issue of choosing the step size in (24) or (31). However for models with \( r < 1/2 \), the numerical differentiation approach may be worthwhile because of the smaller error rate that may be obtained using (24).

Combining (21) and (30), we obtain

\[
\sqrt{n}(\tilde{E}_{\theta|X}(\theta) - \theta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{I}_0^{-1} \tilde{L}_0(X_i) + O_P(M_n(r)).
\]

The range of \( r \) implies that the mean value of the profile sampler is essentially a semiparametric efficient estimator of \( \theta \) even when the nuisance parameter
has a slower convergence rate. A similar conclusion appears to hold for other estimators of $\hat{\theta}_n$ based on the profile sampler, including multivariate generalizations of the median.

The second main result is expressed in the following theorem 4. An $\alpha$-th quantile of the posterior profile distribution is any quantity $\tau_{\alpha} \in \mathbb{R}^d$ that satisfies $\tau_{\alpha} = \inf \{ \xi : \tilde{P}_{\theta|X}(\theta \leq \xi) \geq \alpha \}$, where $\xi$ is an infimum over the given set only if there does not exist a $\xi_1 < \xi$ in $\mathbb{R}^d$ such that $\tilde{P}_{\theta|X}(\theta \leq \xi_1) \geq \alpha$. Because of the assumed smoothness of both the prior and the likelihood in our setting, we can, without loss of generality, assume $\tilde{P}_{\theta|X}(\theta \leq \tau_{\alpha}) = \alpha$. We can also define $\kappa_{\alpha} = \sqrt{n}(\tau_{\alpha} - \hat{\theta}_n)$, i.e., $\tilde{P}_{\theta|X}(\sqrt{n}(\theta - \hat{\theta}_n) \leq \kappa_{\alpha}) = \alpha$. Note that neither $\tau_{\alpha}$ nor $\kappa_{\alpha}$ are unique if the dimension for $\theta$ is larger than one. Nevertheless, the following theorem ensures that for each choice of $\kappa_{\alpha}$ there exists a unique $\hat{\tau}_{\alpha}$ based on the data such that $P(\sqrt{n}(\hat{\theta}_n - \theta_0) \leq \hat{\tau}_{\alpha}) = \alpha$ and $\| \hat{\tau}_{\alpha} - \kappa_{\alpha} \| = O_P(M_n(r))$:

**Theorem 4.** Under the conditions of theorem 3 and assuming that $\tilde{\ell}_0(X)$ has finite third moment with a nondegenerate distribution, then there exists a $\hat{\tau}_{\alpha}$ based on the data such that $P(\sqrt{n}(\hat{\theta}_n - \theta_0) \leq \hat{\tau}_{\alpha}) = \alpha$ and $\hat{\tau}_{\alpha} - \kappa_{\alpha} = O_P(M_n(r))$ for each fixed $\kappa_{\alpha}$.

**Remark 5.** The nondegenerate distribution condition of $\tilde{\ell}_0(X)$ can be satisfied if $X$ has a nonsingular absolutely continuous component. The proof of theorem 4 implies that for any $\hat{\tau}_{\alpha}$ satisfying both $\| \kappa_{\alpha} - \hat{\tau}_{\alpha} \| = O_P(M_n(r))$ and $P(\sqrt{n}(\hat{\theta}_n - \theta_0) \leq \hat{\tau}_{\alpha}) = \alpha$, we have $\| \hat{\tau}_{\alpha} - \kappa_{\alpha} \| = O_P(M_n(r))$. Thus $\hat{\tau}_{\alpha}$ is essentially uniquely determined, at least up to an error of order $O_P(M_n(r))$. 
Remark 6. Clearly, a faster convergence rate of the nuisance parameter leads to a more accurate estimate of the confidence interval when $1/4 < r < 1/2$. The profile sampler procedure can provide the best estimate for the boundary of the confidence interval in semiparametric models when $r \geq 1/2$. We conjecture that the product of $\sqrt{n}I\{r \geq 1/2\} + n^{2r-1/2}I\{1/4 < r < 1/2\}$ and the $O_P(M_n(r))$ term in theorem 4 converges to the product of two different non-trivial but uniformly integrable Gaussian process. Thus we believe the convergence rate in theorem 4 is optimal.

Theorem 4 states that the Wald-type confidence interval can be approximated by the credible set of the same type based on the profile sampler with error of order $O_P(M_n(r))$. In other words, the boundary of a one-sided confidence interval for $\theta$ at level $\alpha$ can be estimated by the $\alpha$-th quantile of the profile sampler with error of order $O_P(n^{-1/2}M_n(r))$. Similar conclusions also hold for the confidence interval obtained by inverting the profile likelihood ratio, as will be shown in theorem 5 below.

The profile likelihood ratio in the frequentist set-up is defined as:

$$PLR_f(\theta_0) = 2(\log p_{l_n}(\hat{\theta}_n) - \log p_{l_n}(\theta_0)).$$

Regarding $\theta$ as a random vector, we define the Bayesian counterpart of the profile likelihood ratio as:

$$PLR_b(\theta) = 2(\log p_{l_n}(\hat{\theta}_n) - \log p_{l_n}(\theta)).$$

Thus $\chi_b^\alpha$ is defined by $\chi_b^{\alpha} = \inf\{\xi : \tilde{F}_{\theta_0} (PLR_b(\theta) \leq \xi) \geq \alpha\}$. As argued previously, we can, without loss of generality, assume that $\tilde{F}_{\theta_0} (PLR_b(\theta) \leq \chi_b^{\alpha}) = \alpha$. The following theorem ensures that there exists a $\chi_f^{\alpha}$ based on the data such that $P(PLR_f(\theta_0) \leq \chi_f^{\alpha}) = \alpha$ and $\chi_f^{\alpha} - \chi_b^{\alpha} = O_P(M_n(r))$:
Theorem 5. Under the conditions of theorem 4, there exists a $\chi_{f}^{na}$ based on the data such that $P(PLR_{f}(\theta_{0}) \leq \chi_{f}^{na}) = \alpha$ and $\chi_{f}^{na} - \chi_{b}^{na} = O_{P}(M_{n}(r))$.

Remark 7. The corresponding $\alpha$-level confidence interval and credible set obtained by inverting the profile likelihood ratio can be expressed as $C_{f}^{na}(X) = \{\theta \in \Theta : PLR_{f}(\theta) \leq \chi_{f}^{na}\}$ and $C_{b}^{na}(X) = \{\theta \in \Theta : PLR_{b}(\theta) \leq \chi_{b}^{na}\}$, respectively. The boundaries of the confidence interval and credible sets of the profile likelihood ratio type are as close as $O_{P}(M_{n}(r))$. Moreover, the proof of theorem 5 implies that $\chi_{b}^{na} = \chi_{d,\alpha}^{2} + O_{P}(M_{n}(r))$ and $\chi_{b}^{na} = \chi_{d,\alpha}^{2} + O_{P}(M_{n}(r))$, where $\chi_{d,\alpha}^{2}$ denotes the $\alpha$-th quantile of central chi-square distribution with degree of freedom $d$. Theorem 5 also implies that $\hat{P}_{\theta|X}(PLR_{b}(\theta) \leq \chi_{d,\alpha}^{2}) = \alpha + O_{P}(M_{n}(r))$. Thus it appears in this instance that not much is gained by using the posterior profile sampler to calibrate the likelihood ratio confidence interval instead of simply using $\chi_{d,\alpha}^{2}$.

5. Examples. This section illustrates the practicality of the stated conditions by verifying that these assumptions are satisfied for each of the three examples introduced in section 2.

5.1. Proportional Hazards Cox Model With Right Censored Data. Note that this example was considered fully in [5], but we include some of the main ideas here for completeness. We first verify the smoothness conditions B1 and the empirical processes assumptions C1. Under regular conditions, B1 can be easily satisfied since the maps $(t, \theta, \eta) \mapsto (\partial^{i+m}/\partial t^{i}\theta^{m})\ell(t, \theta, \eta)$, whose forms can be found in [5], are uniformly bounded around $(\theta_{0}, \theta_{0}, \Lambda_{0})$. Notice that the functions $y \mapsto h_{0}(y), y \mapsto \Lambda_{t}(y)$ and $z \mapsto \exp(zt)$ for $(t, \theta, \Lambda)$
in the assumed neighborhood of the true values are \(P\)-Donsker. Thus we can verify C1 by repeatedly employing the Lipschitz continuity preservation property of Donsker classes. The remaining smoothness conditions B2 and condition (27) are separately verified by lemmas 3 and 4 of [5].

5.2. Proportional Hazards Cox Model With Current Status Data. We can verify that \(\ell(t, \theta, \Lambda)\) defined in section 2.2.2 above is indeed the least favorable submodel since 

\[
\dot{\ell}(t, \theta, \Lambda) = (z\Lambda_t(\theta, \Lambda)(y) - \phi(\Lambda(y))h_{00} \circ \Lambda_{0}^{-1} \circ \Lambda(y))Q(x; t, \Lambda_t(\theta, \Lambda)),
\]

evaluated at \(t = \theta = \theta_0\) and \(\Lambda = \Lambda_0\), is the efficient score function \((z\Lambda_0(y) - h_0(y))Q(x; \theta_0, \Lambda_0)\). Note that we extend the domain of the function \(u \mapsto \Lambda_0^{-1}(u)\) to all of \([0, \infty)\) by assigning the value \(\sigma\) to all \(u \in [0, \Lambda(\sigma)]\) and the value \(\tau\) to all \(u > \Lambda(\tau)\). Substituting \(\theta = t\) and \(\Lambda = \Lambda_t(\theta, \Lambda)\) in (8) and differentiating with respect to \(t\) and \(\theta\), we obtain,

\[
\dot{\ell}(t, \theta, \Lambda)(x) = (z\Lambda_t + \dot{\Lambda}_t)Q(x; t, \Lambda_t),
\]

\[
\ddot{\ell}(t, \theta, \Lambda)(x) = \frac{\partial^2 \text{lik}(t, \Lambda_t(\theta, \Lambda))}{l\text{ik}(t, \Lambda_t(\theta, \Lambda))} - \dot{\ell}^2(t, \theta, \Lambda),
\]

\[
\ell_{t, \theta}(t, \theta, \Lambda)(x) = (z\Lambda_t + \dot{\Lambda}_t)\dot{\Lambda}_t \exp(-e^{tx}\Lambda_t) \left(\frac{\delta e^{tx}}{1 - \exp(-e^{tx}\Lambda_t)}\right)^2 - zQ(x; t, \Lambda_t)\dot{\Lambda}_t,
\]

\[
\ell^{(3)}(t, \theta, \Lambda)(x) = \frac{\partial}{\partial t} \left(\frac{\partial^2 \text{lik}(t, \Lambda_t(\theta, \Lambda))}{l\text{ik}(t, \Lambda_t(\theta, \Lambda))}\right) - 2\dot{\ell}(t, \theta, \Lambda)\ddot{\ell}(t, \theta, \Lambda),
\]

\[
\ell_{t, t, \theta}(t, \theta, \Lambda)(x) = \frac{\partial}{\partial \theta} \left(\frac{\partial^2 \text{lik}(t, \Lambda_t(\theta, \Lambda))}{l\text{ik}(t, \Lambda_t(\theta, \Lambda))}\right) - 2\dot{\ell}(t, \theta, \Lambda)\ell_{t, \theta}(t, \theta, \Lambda),
\]

\[
\ell_{t, \theta, \theta}(t, \theta, \Lambda)(x) = \delta e^{3tx}\dot{\Lambda}_t^2(z\Lambda_t + \dot{\Lambda}_t) \times \frac{\exp(-e^{tx}\Lambda_t)(1 + \exp(-e^{tx}\Lambda_t))}{(1 - \exp(-e^{tx}\Lambda_t))^3} - 2\delta ze^{2tx}\dot{\Lambda}_t^2 \times \frac{\exp(-e^{tx}\Lambda_t)}{(1 - \exp(-e^{tx}\Lambda_t))^2},
\]
where

\[ \Lambda_t \equiv \Lambda_t(\theta, \Lambda), \]

\[ \dot{\Lambda}_t \equiv \partial \Lambda_t / \partial t = -\phi(\Lambda)(y) h_{00} \circ \Lambda_0^{-1} \circ \Lambda(y), \]

\[ N(t, \theta, \Lambda) \equiv \frac{\partial^2 \text{lik}(t, \Lambda_t(\theta, \Lambda)) / \partial t^2}{\text{lik}(t, \Lambda_t(\theta, \Lambda))} = Q(x; t, \Lambda_t) \times \left[ z^2 \Lambda_t + 2 \dot{\Lambda}_t - e^{tz}(z \Lambda_t + \dot{\Lambda}_t)^2 \right], \]

\[ \frac{\partial}{\partial t} (N(t, \theta, \Lambda)) = \]

\[ Q(x; t, \Lambda_t) \times \left( z^2 \Lambda_t - z e^{tz}(z \Lambda_t + \dot{\Lambda}_t)(z \Lambda_t + 3 \dot{\Lambda}_t) \right) + (zQ(x; t, \Lambda_t) - e^{tz}(z \Lambda_t + \dot{\Lambda}_t)^2), \]

\[ \frac{\partial}{\partial \theta} (N(t, \theta, \Lambda)) = \]

\[ \left( \frac{e^{2tz} \dot{\Lambda}_t \exp(-e^{tz} \Lambda_t)}{(1 - \exp(-e^{tz} \Lambda_t))^2} \right) \times \left( z^2 \Lambda_t + 2 \dot{\Lambda}_t - e^{tz}(z \Lambda_t + \dot{\Lambda}_t)^2 \right) - Q(x; t, \Lambda_t) \times \]

\[ (z^2 \Lambda_t - 2 e^{tz}(z \Lambda_t + \dot{\Lambda}_t)) \dot{\Lambda}_t. \]

We next show that the maps \((t, \theta, \Lambda) \mapsto \partial t^i / \partial t^j \partial \theta^m \ell(t, \theta, \Lambda)\) are uniformly bounded for all \(t\) sufficiently close to \(\theta\) and all \(\Lambda\) in the parameter space, for \((l, m) = (0, 0), (1, 0), (2, 0), (3, 0), (1, 1), (1, 2), (2, 1)\). Note that \(\dot{\ell}(t, \theta, \Lambda)\) can be written as follows:

\[ \dot{\ell}(t, \theta, \Lambda) = \left[ z - \frac{\phi(\Lambda)(y)}{\Lambda_t(\theta, \Lambda)} h_{00} \circ \Lambda_0^{-1} \circ \Lambda(y) \right] \Lambda_t(\theta, \Lambda)(y) Q(x; t, \Lambda_t(\theta, \Lambda)), \]

and the map \(u \mapsto ue^{-u}/(1 - e^{-u})\) is bounded and Lipschitz on \([0, \infty)\). Thus we can write \(\Lambda(y)Q(x; t, \Lambda) = \psi(e^{tz}, \Lambda(y))\), where the function \(\psi\) is bounded and Lipschitz in each argument. Next, note that

\[ \frac{\dot{\Lambda}_t}{\Lambda_t} - \frac{\phi(\Lambda) h_{00} \circ \Lambda_0^{-1} \circ \Lambda}{\Lambda_t(\theta, \Lambda)} = \frac{(\phi(\Lambda)/\Lambda) h_{00} \circ \Lambda_0^{-1} \circ \Lambda}{1 + (\theta - t)(\phi(\Lambda)/\Lambda) h_{00} \circ \Lambda_0^{-1} \circ \Lambda}. \]
Combining this with the facts that the function \( \phi(y)/y \) is bounded and 
\( h_{00} \circ \Lambda_0^{-1} \) is bounded by assumption, we obtain that 
\( \ell(t, \theta, \Lambda) \) is bounded. 
Clearly, \( \ell(t, \theta, \Lambda) \) is also uniformly bounded based on the following equation:

\[
\frac{\partial^2 \tilde{\ell}(t, \Lambda_t(\theta, \Lambda))}{\partial t^2} = Q(x; t, \Lambda_t) \Lambda_t \\
\times \left( z^2 + 2 \frac{\Lambda_t}{\Lambda} - e^{tz} \Lambda_t \left( z^2 + 2z \frac{\dot{\Lambda}_t}{\Lambda} + \left( \frac{\ddot{\Lambda}_t}{\Lambda} \right)^2 \right) \right).
\]

By similar analysis, \( \ell_t(\theta, \Lambda), \ell^{(3)}(t, \theta, \Lambda), \ell_{t,t}(\theta, \Lambda) \) and \( \ell_{t,\theta,\theta}(t, \theta, \Lambda) \) 
are also uniformly bounded for all \( t \) sufficiently close to \( \theta \) and all \( \Lambda \) varying over the parameter space.

We next verify assumption C1. Recall that \( \Lambda(y)Q(x; t, \Lambda) = \psi(e^{tx}, \Lambda(y)) \),
where the function \( \psi \) is bounded and Lipschitz in each argument. Thus, 
since the classes of functions \( z \mapsto e^{tx} \) and \( y \mapsto \Lambda(y) \) are Donsker, 
so is the class of functions \( x \mapsto \Lambda(y)Q(x; t, \Lambda) \). Note that

\[
\frac{\phi(\Lambda)}{\Lambda_t(\theta, \Lambda)} = \frac{\varsigma(\Lambda)}{1 + (\theta - t)\varsigma(\Lambda)\upsilon(\Lambda)} \equiv \chi(\Lambda),
\]

where \( \varsigma(\Lambda) = \phi(\Lambda)/\Lambda \) and \( \upsilon(\Lambda) = h_{00} \circ \Lambda_0^{-1} \circ \Lambda \), 
and both \( \varsigma(\Lambda) \) and \( \upsilon(\Lambda) \) are Lipschitz according to the assumptions. Hence \( \chi(\Lambda) \) is also Lipschitz in \( \Lambda \).
Thus the class of functions \( \hat{\ell}(t, \theta, \Lambda) \) with \((t, \theta)\) varying over a small neighborhood of \((\theta_0, \theta_0)\) and \( \Lambda \) ranging over all nondecreasing cadlag functions with domain \([\sigma, \tau]\) and range \([0, M]\) can be seen to be a Donsker class. By repeated application of the above techniques to \( \langle \partial^2 \tilde{\ell}(t, \Lambda_t(\theta, \Lambda)) / \partial t^2 \rangle / \tilde{\ell}(t, \Lambda_t(\theta, \Lambda)) \)
we know the class of functions \( \tilde{\ell}(t, \theta, \Lambda) \) is also Donsker. Similarly, the classes of functions \( \ell_t(\theta, \Lambda) \) and \( \ell^{(3)}(t, \theta, \Lambda) \) with \((t, \theta)\) varying over a small neighborhood of \((\theta_0, \theta_0)\) and \( \Lambda \) ranging over all nondecreasing cadlag functions with domain \([\sigma, \tau]\) and range \([0, M]\) can be shown to be Donsker. More-
over, \( \ell^{(3)}(t, \theta, \Lambda) \) is automatically P-Glivenko-Cantelli since it is uniformly bounded based on the previous analysis.

The following lemmas verify the remaining assumptions:

**Lemma 2.** Under the above set-up for the Cox model with current status data, assumption B2 is satisfied.

**Lemma 3.** Under the above set-up for the Cox model with current status data, condition (27) is satisfied.

5.3. Partly Linear Normal Model With Current Status Data. By differentiating the the least favorable submodel (12) with respect to \( t \) or \( \theta \), we can obtain

\[
\begin{align*}
\ell(t, \theta, k) &= Q(x; t, k_t)(w - h_0(z)) , \\
\ell^2(t, \theta, k) &= (w - h_0(z))^2 \phi_t \left[ (1 - \delta) \frac{(1 - \Phi_t) q_t - \phi_t}{(1 - \Phi_t)^2} - \delta q_t \Phi_t + \phi_t \right] , \\
\ell_{t, \theta}(t, \theta, k) &= (w - h_0(z)) h_0(z) \phi_t \left[ (1 - \delta) \frac{(1 - \Phi_t) q_t - \phi_t}{(1 - \Phi_t)^2} - \delta q_t \Phi_t + \phi_t \right] , \\
\ell^{(3)}(t, \theta, k) &= (w - h_0(z))^3 \phi_t R(q_t(x)) , \\
\ell_{t, \theta, \theta}(t, \theta, k) &= (w - h_0(z))^2 h_0(z) \phi_t R(q_t(x)) , \\
\ell_{t, \theta, \theta, \theta}(t, \theta, k) &= (w - h_0(z)) h_0^2(z) \phi_t R(q_t(x)), \end{align*}
\]

where

\[
R(q_t(x)) = \left[ (1 - \delta) \left( \frac{q_t^2 - 1}{1 - \Phi_t} + \frac{\phi_t q_t^2 - 2\phi_t q_t}{(1 - \Phi_t)^2} + \frac{2\phi_t^2}{(1 - \Phi_t)^3} \right) \right.
- \delta \left( \frac{q_t^2 - 1}{\Phi_t} + \frac{3\phi_t q_t}{\Phi_t^2} + \frac{2\phi_t^2}{\Phi_t^3} \right) ,
\]

\( q_t = q_{t, k_t}(\theta, k)(x) , \phi_t = \phi(q_t) , \) and \( \Phi_t = \Phi(q_t) \). The convergence rate for the estimated nuisance parameter is established in lemma 4 by application of theorem 1. The rate \( r = 2/5 \) is clearly faster than the cubic rate but slower than the parametric rate. Note that \( \mathcal{O}_2^M \) is a P-Donsker class by
technical tool T4 in the appendix. Assumption C1 can be verified easily by recognizing that the three classes of functions specified in C1 depend on $(t, \theta, k)$ in a Lipschitz manner and are uniformly bounded. The remaining assumptions are verified in lemmas 5 and lemma 6 below.

**Lemma 4.** Under the above set-up for the partly linear normal model with current status data, we have

$$\|k_{\hat{\theta}_n} - k_0\|_{L_2} = O_P(n^{-2/5} + \|\hat{\theta}_n - \theta_0\|),$$

for $\hat{\theta}_n \overset{P}{\rightarrow} \theta_0$.

**Lemma 5.** Under the above set-up for the partly linear normal model with current status data, assumptions B1 and B2 are satisfied.

**Lemma 6.** Under the above set-up for the partly linear normal model with current status data, condition (27) is satisfied.

6. **Future Work.** It is clear that the estimation accuracy for $\theta$ in the profile sampler method is intrinsically determined by the semiparametric model specifications, specifically by the convergence rate of the nuisance parameter. Therefore it is very natural to raise a question about how to control the degree of accuracy. We now propose two potential strategies as future topics. One approach is to construct the fully Bayesian procedure, which assigns a prior on both the parameter of interest and on the functional nuisance parameter rather than simply profiling the nuisance parameter out. This strategy has been studied in [20] by considering the marginal posterior distribution of $\theta$ in terms of its first order frequentist properties. However, to compare with the profile sampler method, we need to figure out the
corresponding second order frequentist properties. Another strategy is to profile the penalized likelihood, whose penalty term is some norm on the nuisance parameter space such as the Sobolev norm. We expect that we can adjust the estimation accuracy of the proposed penalized profile sampler by tuning the corresponding smoothing parameter.

We believe that under certain special model specifications, third or higher order semiparametric frequentist inference can be constructed by extending the Bartlett correction [1] and objective prior [25] results to semiparametric settings. There is a rich literature on the higher order properties of posteriors for parametric models and the choice of the prior; see, for example, [10], [17], [18], [19], [21] and [22]. The detailed comparisons between the two approaches to estimate \( \hat{I}_0 \) need to be studied formally in the future. Given a class of loss functions, we can also investigate the frequentist properties of the Bayes estimator \( T_n \) for \( \theta_0 \) under the posterior profile distribution.

7. Appendix. Proof of theorem 2: We first prove (21). Note that

\[
0 = \mathbb{P}_n \hat{\epsilon}(\hat{\theta}_n, \hat{\eta}_n) = \mathbb{P}_n \hat{\epsilon}(\theta_0, \hat{\theta}_n, \hat{\eta}_n) + \mathbb{P}_n \hat{\epsilon}(\theta_0, \hat{\theta}_n, \hat{\eta}_n)(\hat{\theta}_n - \theta_0) + \frac{1}{2} (\hat{\theta}_n - \theta_0)^T \otimes \mathbb{P}_n \epsilon(3)(\theta_0, \hat{\theta}_n, \hat{\eta}_n) \otimes (\hat{\theta}_n - \theta_0),
\]

where \( \hat{\theta}_n \) is in between \( \theta_0 \) and \( \hat{\theta}_n \). By considering lemma 7 and lemma 8 below, we can derive the following:

\[
0 = n^{-1} \sum_{i=1}^{n} \tilde{\ell}_0(x_i) + P \tilde{\ell}_0(\hat{\theta}_n - \theta_0) + n^{-1/2} \tilde{I}_0 \Delta_{4n}(\theta_0, \hat{\theta}_n, \hat{\eta}_n), \text{ and}
\]

\[
(34) \quad \sqrt{n}(\hat{\theta}_n - \theta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \tilde{I}_0^{-1} \tilde{\ell}_0(X_i) + \Delta_{4n}(\theta_0, \hat{\theta}_n, \hat{\eta}_n),
\]

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where

\[
\Delta_{4n}(\theta_0, \hat{\theta}_n, \eta_n) = \sqrt{n}I_0^{-1}\Delta_{1n}(\theta_0, \hat{\theta}_n, \eta_n) + \sqrt{n}I_0^{-1}\Delta_{2n}(\theta_0, \hat{\theta}_n, \eta_n)(\hat{\theta}_n - \theta_0) \\
+ \frac{1}{2}\sqrt{n}I_0^{-1}(\hat{\theta}_n - \theta_0)^T \otimes \mathbb{E}_{\eta_n}(\partial^3)(\theta_0, \hat{\theta}_n, \eta_n) \otimes (\hat{\theta}_n - \theta_0)
\]

and \(\Delta_{1n}\) and \(\Delta_{2n}\) are defined in the proofs of lemmas 7 and 8, respectively. The orders of magnitude of \(\Delta_{1n}(\theta_0, \hat{\theta}_n, \eta_n)\) and \(\Delta_{2n}(\theta_0, \hat{\theta}_n, \eta_n)\) obtained in the proofs of lemmas 7 and lemma 8 imply that the order of magnitude of \(\Delta_{4n}(\theta_0, \hat{\theta}_n, \eta_n)\) is \(O_P(M_n(r))\), as desired.

We next show (22). By lemma 9 below, we have

\[
\log p_l_n(\hat{\theta}_n) = \log p_l_n(\theta_0) + (\hat{\theta}_n - \theta_0)^T \sum_{i=1}^{n} \tilde{G}_0(X_i) - \frac{n}{2}(\hat{\theta}_n - \theta_0)^T I_0(\hat{\theta}_n - \theta_0) \\
+ \Delta_{3n}(\theta_0, \hat{\theta}_n, \eta_n).
\]

Lemma 9 also implies that

\[
\log p_l_n(\hat{\theta}_n) = \\
\log p_l_n(\hat{\theta}_n) + (\hat{\theta}_n - \hat{\theta}_n)^T \left( \sum_{i=1}^{n} \tilde{G}_0(X_i) - nI_0(\hat{\theta}_n - \theta_0) \right) \\
- \frac{n}{2}(\hat{\theta}_n - \hat{\theta}_n)^T I_0(\hat{\theta}_n - \hat{\theta}_n) + \Delta_{3n}(\theta_0, \hat{\theta}_n, \eta_{\hat{\theta}_n}) - \Delta_{3n}(\theta_0, \hat{\theta}_n, \eta_{\hat{\theta}_n}).
\]

Define \(\Delta_{5n}(\tilde{\theta}_n, \hat{\theta}_n) = \log p_l_n(\tilde{\theta}_n) - \log p_l_n(\hat{\theta}_n) + (n/2)(\tilde{\theta}_n - \hat{\theta}_n)^T I_0(\tilde{\theta}_n - \hat{\theta}_n).

By considering (34), we can obtain the respective upper and lower bounds of \(\Delta_{5n}(\tilde{\theta}_n, \hat{\theta}_n)\) as follows:

\[
\Delta_{5n}^U = -\sqrt{n}(\tilde{\theta}_n - \hat{\theta}_n)^T I_0\Delta_{4n}(\theta_0, \tilde{\theta}_n, \eta_n) + \Delta_{5n}^U(\theta_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) - \Delta_{5n}^U(\theta_0, \hat{\theta}_n, \eta_{\hat{\theta}_n}),
\]

\[
\Delta_{5n}^L = -\sqrt{n}(\tilde{\theta}_n - \hat{\theta}_n)^T I_0\Delta_{4n}(\theta_0, \tilde{\theta}_n, \eta_n) + \Delta_{5n}^L(\theta_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) - \Delta_{5n}^L(\theta_0, \hat{\theta}_n, \eta_{\hat{\theta}_n}),
\]

where \(\Delta_{5n}^L\) and \(\Delta_{5n}^U\) are defined in the proof of lemma 9 and also shown to have magnitude \(O_P(g_r(\|\tilde{\theta}_n - \hat{\theta}_n\|))\). Now the assumptions in section 2 imply that \(\Delta_{5n}^U(\tilde{\theta}_n, \hat{\theta}_n)\) and \(\Delta_{5n}^L(\tilde{\theta}_n, \hat{\theta}_n)\) are of order \(O_P(g_r(\|\tilde{\theta}_n - \hat{\theta}_n\|_2))\), and the proof is complete. □
Lemma 7. Assuming the conditions of theorem 2, we have

\[ P(\hat{\theta}_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) = O_P(M_n(r)n^{-1/2} + \|\tilde{\theta}_n - \hat{\theta}_n\|^2), \]

\[ P_n(\hat{\theta}_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) = P_n(\tilde{\theta}_0 + O_P(M_n(r)) + \|\tilde{\theta}_n - \hat{\theta}_n\|^2), \]

when \( \tilde{\theta}_n - \hat{\theta}_n \xrightarrow{P} 0 \).

Proof. By Taylor expansion of \( \theta \mapsto P(\hat{\theta}_0, \theta, \eta_{\tilde{\theta}_n}) \), we obtain:

\[
P(\hat{\theta}_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) = P(\hat{\theta}_0, \theta_0, \eta_{\tilde{\theta}_n})(\tilde{\theta}_n - \theta_0) \\
+ \frac{1}{2}(\tilde{\theta}_n - \theta_0)^T \otimes P_\ell(\tilde{\theta}_n, \eta_{\tilde{\theta}_n} \otimes (\tilde{\theta}_n - \theta_0) \\
= \Delta_1(\theta_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}),
\]

where \( \theta_0^* \) is intermediate between \( \tilde{\theta}_n \) and \( \theta_0 \). The assumptions in section 2 imply \( \Delta_1(\theta_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) \) has order \( O_P(M_n(r)) \). By writing \( \mathbb{G}_n(\hat{\ell}(\theta_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) - \tilde{\ell}_0) \) as the summation of \( \mathbb{G}_n(\hat{\ell}(\theta_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) - \hat{\ell}(\theta_0, \theta_0, \eta_{\tilde{\theta}_n})) \) and \( \mathbb{G}_n(\hat{\ell}(\theta_0, \theta_0, \eta_{\tilde{\theta}_n}) - \tilde{\ell}_0) \), we obtain that the difference between \( P_n(\hat{\theta}_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) \) and \( P_n(\tilde{\theta}_0) \) is

\[
\Delta_1(\theta_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) = \Delta_1(\theta_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) + n^{-1/2}\mathbb{G}_n(\hat{\ell}(\tilde{\theta}_n, \eta_{\tilde{\theta}_n} - \tilde{\ell}_0),
\]

where \( \theta_2^* \) is intermediate between \( \tilde{\theta}_n \) and \( \theta_0 \). The order of magnitude of \( \Delta_1(\theta_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) \) follows from the assumptions. This completes the proof. □

Lemma 8. Assuming the conditions of theorem 2, we have

\[ P(\hat{\theta}_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) = P(\tilde{\theta}_0 + O_P(M_n(r)) + \|\tilde{\theta}_n - \hat{\theta}_n\|), \]

\[ P_n(\hat{\theta}_0, \tilde{\theta}_n, \eta_{\tilde{\theta}_n}) = P(\tilde{\theta}_0 + O_P(M_n(r)) + \|\tilde{\theta}_n - \hat{\theta}_n\|), \]

for \( \tilde{\theta}_n - \hat{\theta}_n \xrightarrow{P} 0 \).
Proof. By analysis similar to the proof of lemma 7, we obtain

\[ P\tilde{\ell}(\theta_0, \tilde{\theta}_n, \tilde{\eta}_n) = P\tilde{\ell}_0 + \Delta_2(\theta_0, \tilde{\theta}_n, \tilde{\eta}_n), \text{ and} \]
\[ \mathbb{P}_n\tilde{\ell}(\theta_0, \tilde{\theta}_n, \tilde{\eta}_n) = P\tilde{\ell}_0 + \Delta_{2n}(\theta_0, \tilde{\theta}_n, \tilde{\eta}_n), \]

where \( \Delta_2(\theta_0, \tilde{\theta}_n, \tilde{\eta}_n) = (\tilde{\theta}_n - \theta_0)^T \otimes P\ell_t, t(\theta_0, \theta_3, \tilde{\eta}_n) + \left[ P\tilde{\ell}(\theta_0, \theta_0, \tilde{\eta}_n) - P\tilde{\ell}_0 \right] \)
and \( \Delta_{2n}(\theta_0, \tilde{\theta}_n, \tilde{\eta}_n) = \Delta_2(\theta_0, \tilde{\theta}_n, \tilde{\eta}_n) + n^{-1/2} \mathbb{G}_n\ell(\theta_0, \tilde{\theta}_n, \tilde{\eta}_n), \) and where \( \theta_3 \) is in between \( \theta_0 \) and \( \tilde{\theta}_n \). The assumptions in section 2 now yield the desired order of magnitude in (38) and (39). \( \square \)

Lemma 9. Assuming the conditions of theorem 2, we have

\[ \log p_{\tilde{\theta}_n} = \log p_{\tilde{\theta}_n}^{\|} + (\tilde{\theta}_n - \theta_0)^T \sum_{i=1}^{\mathbb{P}_n} \tilde{\ell}_0(X_i) \]
\[ \frac{-n}{2}(\tilde{\theta}_n - \theta_0)^T \tilde{I}_0(\tilde{\theta}_n - \theta_0) + O_P(\|\tilde{\theta}_n - \tilde{\theta}_n\|), \]

for \( \tilde{\theta}_n - \hat{\theta}_n \xrightarrow{p} 0 \).

Proof. \( n^{-1} \left( \log p_{\tilde{\theta}_n} - \log p_{\tilde{\theta}_n}^{\|} \right) = \mathbb{P}_n\ell(\tilde{\theta}_n, \tilde{\theta}_n, \tilde{\eta}_n) - \mathbb{P}_n\ell(\theta_0, \theta_0, \tilde{\eta}_0). \)

The right hand side of the above equation is bounded below and above by \( \mathbb{P}_n(\ell(\tilde{\theta}_n, \tilde{\theta}_n) - \ell(\theta_0, \tilde{\theta}_n)), \) where the lower and upper bounds separately correspond to \( \tilde{\theta}_n = (\theta_0, \tilde{\theta}_n) \) and \( (\tilde{\theta}_n, \tilde{\eta}_n) \). By applying a three term Taylor expansion to both upper and lower bounds, we obtain the the corresponding upper bound, \( \Delta_{3n}^{\|}(\theta_0, \tilde{\theta}_n, \tilde{\eta}_n) \), and lower bound, \( \Delta_{3n}(\theta_0, \tilde{\theta}_n, \tilde{\eta}_n) \), for \( \Delta_{3n}(\theta_0, \tilde{\theta}_n, \tilde{\eta}_n) \) defined as follows:

\[ \Delta_{3n}(\theta_0, \tilde{\theta}_n, \tilde{\eta}_n) \equiv \log p_{\tilde{\theta}_n} - \log p_{\tilde{\theta}_n}^{\|} = (\tilde{\theta}_n - \theta_0)^T \sum_{i=1}^{\mathbb{P}_n} \tilde{\ell}_0(X_i) \]
\[ + \frac{1}{2}n(\tilde{\theta}_n - \theta_0)^T \tilde{I}_0(\tilde{\theta}_n - \theta_0), \]
where

\[
\Delta^U_{3n}(\theta_0, \hat{\theta}_n, \eta_{\theta_n}) = n(\hat{\theta}_n - \theta_0)^T \Delta_{1n}(\theta_0, \hat{\theta}_n, \eta_{\theta_n}) + \frac{n}{2}(\hat{\theta}_n - \theta_0)^T \Delta_{2n}(\theta_0, \hat{\theta}_n, \eta_{\theta_n})
\]

\[
\times (\hat{\theta}_n - \theta_0) + \frac{n}{6} \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} \mathbb{P}_{\theta_i,t_j,t_k} (\theta^*_i, \hat{\theta}_n, \eta_{\theta_n}) (\hat{\theta}_n - \theta_0)_i (\hat{\theta}_n - \theta_0)_j (\hat{\theta}_n - \theta_0)_k,
\]

\[
\Delta^L_{3n}(\theta_0, \hat{\theta}_n, \eta_{\theta_n}) = n(\hat{\theta}_n - \theta_0)^T \Delta_{1n}(\theta_0, \theta_0, \hat{\eta}_0) + \frac{n}{2}(\hat{\theta}_n - \theta_0)^T \Delta_{2n}(\theta_0, \theta_0, \hat{\eta}_0)
\]

\[
\times (\hat{\theta}_n - \theta_0) + \frac{n}{6} \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} \mathbb{P}_{\theta_i,t_j,t_k} (\theta^*_i, \theta_0, \hat{\eta}_0) (\hat{\theta}_n - \theta_0)_i (\hat{\theta}_n - \theta_0)_j (\hat{\theta}_n - \theta_0)_k,
\]

where \(\theta^*_i\) and \(\theta^*_i\) are in between \(\theta_0\) and \(\hat{\theta}_n\). Lemma 7 and lemma 8 yield the order of \(\Delta^U_{3n}(\theta_0, \hat{\theta}_n, \eta_{\theta_n})\) and \(\Delta^L_{3n}(\theta_0, \hat{\theta}_n, \eta_{\theta_n})\), which is \(O_P(\|\hat{\theta}_n - \theta_0\|)\). \(\square\)

**Proof of theorem 3.** Suppose that \(F_n(\cdot)\) is the posterior profile distribution of \(\sqrt{n} \hat{\theta}_n\) with respect to the the prior \(\rho(\theta)\), where the vector \(\varrho_n\) is defined as \(\hat{\theta}_n^{1/2}(\theta - \hat{\theta}_n)\). Let the parameter set for \(\varrho_n\) be \(\Xi_n\). The whole proof of theorem 3 can be briefly summarized in the following expression:

\[
F_n(\xi) = \int_{\Xi_n \cap \Xi_{n} \cap \Xi_{n} \cap \Xi_{n}} \rho(\hat{\theta}_n + \hat{I}_0^{1/2} \varrho_n) \frac{\mathbb{P}_{\theta_n}(\varrho_n + \hat{I}_0^{-1/2} \varrho_n)}{\mathbb{P}_{\theta_n}(\varrho_n)} d\varrho_n.
\]

Note that \(d\varrho_n\) above is the short notation for \(d\varrho_{n1} \times \ldots \times d\varrho_{nd}\). We first partition the parameter set \(\Xi_n\) as \(\{\Xi_n \cap \{\|\varrho_n\| > r_n\}\} \cup \{\Xi_n \cap \{\|\varrho_n\| \leq r_n\}\}\}. By choosing the proper order of \(r_n\), we find the posterior mass in the first partition is of arbitrarily small order and the mass inside the second partition region can be approximated by a stochastic polynomial in powers of \(n^{-1/2}\) with an error of order dependent on the convergence rate of the nuisance parameter. This general approach applies to both the denominator and numerator, yielding a quotient series that leads to the desired result.

Before giving the formal proof, we need two intermediate lemmas:
Lemma 3.1. Choose \( r_n = o(n^{-1/3}) \) such that \( \sqrt{n}r_n \to \infty \). Under the conditions of theorem 3, we have

\[
\int_{\|\phi_n\|>r_n} \rho(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \phi_n) \frac{p_{l_n}(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \phi_n)}{p_{l_n}(\hat{\theta}_n)} d\phi_n = O_P(n^{-M}),
\]

for any positive number \( M \).

Proof. Fix \( r > 0 \). We have

\[
\int_{\|\phi_n\|>r} \rho(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \phi_n) \frac{p_{l_n}(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \phi_n)}{p_{l_n}(\hat{\theta}_n)} d\phi_n \\
\leq I\{\Delta_n^r < -n^{-\frac{1}{2}}\} \exp(-\sqrt{n}) \int_{\Theta} \rho(\theta) d\theta + I\{\Delta_n^r \geq -n^{-\frac{1}{2}}\},
\]

where \( \Delta_n^r = \sup_{\|\phi_n\|>r} \Delta_n(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \phi_n) \). By lemma 3.2 in [5], \( I\{\Delta_n^r \geq -n^{-\frac{1}{2}}\} = O_P(n^{-M}) \) for any fixed \( r > 0 \). This implies that there exists a positive decreasing sequence \( r_n = o(n^{-1/3}) \) with \( \sqrt{n}r_n \to \infty \) such that (41) holds.

Lemma 3.2. Choose \( r_n = o(n^{-1/3}) \) such \( \sqrt{n}r_n \to \infty \). Under the conditions of theorem 3, we have

\[
\int_{\|\phi_n\| \leq r_n} \left| \frac{p_{l_n}(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \phi_n)}{p_{l_n}(\hat{\theta}_n)} - \rho(\hat{\theta}_n) \right| \rho(\phi_n) d\phi_n = O_P(n^{-1/2} M_n(r)).
\]

Proof. The posterior mass over the region \( \|\phi_n\| \leq r_n \) is bounded by

\[
\int_{\|\phi_n\| \leq r_n} \left| \frac{p_{l_n}(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \phi_n)}{p_{l_n}(\hat{\theta}_n)} - \rho(\hat{\theta}_n) \right| \rho(\phi_n) d\phi_n (*),
\]

\[
+ \int_{\|\phi_n\| \leq r_n} \left| \frac{p_{l_n}(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \phi_n)}{p_{l_n}(\hat{\theta}_n)} - \frac{p_{l_n}(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \phi_n)}{p_{l_n}(\hat{\theta}_n)} \rho(\hat{\theta}_n) \right| \rho(\phi_n) d\phi_n. (**)
\]
Using (22) when $r \geq 1/2$, we obtain

\[
(*) = \int_{\|\varrho_n\| \leq r_n} \left[ \rho(\hat{\theta}_n) \exp \left( -\frac{n \varrho_n^T \varrho_n}{2} \right) \exp(O_P(n\|\varrho_n\|^3 + n^{-\frac{3}{2}})) - 1 \right] d\varrho_n \\
= n^{-\frac{1}{2}} \int_{\|u_n\| \leq \sqrt{n}r_n} \left[ \rho(\hat{\theta}_n) \exp \left( -\frac{u_n^T u_n}{2} \right) \exp(n^{-\frac{1}{2}}(\|u_n\|^3 + 1)O_P(1)) - 1 \right] du_n \\
= n^{-1} \times O_P(1) \times \int_{\|u_n\| \leq \sqrt{n}r_n} \left[ \rho(\hat{\theta}_n) \exp \left( -\frac{u_n^T u_n}{2} \right) (\|u_n\|^3 + 1) \right] du_n \\
= O_P(n^{-1}),
\]

where the second equality follows by replacing $\sqrt{n}\varrho_n$ with $u_n$, and the third equality follows from the fact that $|\exp(n^{-1/2}(\|u_n\|^3 + 1)O_P(1)) - 1| = O_P(1)n^{-1/2}(\|u_n\|^3 + 1)$, since $n^{-1/2}\|u_n\|^3 = o(1)$.

However, when $1/4 < r < 1/2$, we obtain

\[
(*) = \int_{\|\varrho_n\| \leq r_n} \left[ \rho(\hat{\theta}_n) \exp \left( -\frac{n \varrho_n^T \varrho_n}{2} \right) \exp(O_P(g_r(\|\varrho_n\|))) - 1 \right] d\varrho_n \\
= \int_{\|\varrho_n\| \leq r_n} \left[ \rho(\hat{\theta}_n) \exp \left( -\frac{n \varrho_n^T \varrho_n}{2} \right) g_r(\|\varrho_n\|) \times O_P(1) \right] d\varrho_n.
\]

When $1/4 < r < 1/3$, $O_P(g_r(\|\varrho_n\|)) = O_P(n^{1-2r}\|\varrho_n\| + n^{-2r+1/2})$. Note that there exists a $\delta > 0$ such that $r_n = n^{2r-1-\delta}$ satisfying $r_n = o(n^{-1/3})$ with $\sqrt{n}r_n \to \infty$ for any $1/4 < r < 1/3$. Therefore $n^{1-2r}\|\varrho_n\| + n^{-2r+1/2} = o(1)$ when $r_n$ is taken equal to $n^{2r-1-\delta}$ for some $\delta > 0$. In this case, it implies that $\tilde{g}_r(\|\varrho_n\|) = n^{1-2r}\|\varrho_n\| + n^{-2r+1/2}$ for $1/4 < r < 1/3$. However $\tilde{g}_r(\|\varrho_n\|) = g_r(\|\varrho_n\|)$ for $1/3 \leq r < 1/2$ since $g_r(\|\varrho_n\|) = o(1)$ when $r$ is in this range. Combining with previous analyses, we have $(*) = O_P(n^{-2r})$ for $1/4 < r < 1/2$. Summarizing the above analysis, we have $(*) = O_P(n^{-1/2}M_n(r))$.

By the following analysis of $(**)$, we will be able to show that $(**)$ = $O_P(n^{-1})$ for $r \geq 1/2$ since $\exp(O_P(n\|\varrho_n\|^3 + n^{-1/2})) = O_P(1)$ with $\|\varrho_n\| \leq$
for $r_n$: 
\[
(\ast \ast) = \int_{\|\varrho_n\| \leq r_n} \left[ \rho(\theta_n^*)^T I_0^{-\frac{1}{2}} \varrho_n \exp \left( -\frac{n}{2} \varrho_n^T \varrho_n + O_P(n\|\varrho_n\|^3 + n^{-\frac{1}{2}}) \right) \right] d\varrho_n \\
\leq M \int_{\|\varrho_n\| \leq r_n} \left[ \|\varrho_n\| \exp \left( -\frac{n}{2} \varrho_n^T \varrho_n \right) \right] d\varrho_n \times \sup_{\|\varrho_n\| \leq r_n} \exp \left( O_P(n\|\varrho_n\|^3 + n^{-\frac{1}{2}}) \right),
\]
where $\theta_n^*$ is intermediate between $\hat{\theta}_n$ and $\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \varrho_n$.

In the case that $1/4 < r < 1/2$, we have the same conclusion:
\[
(\ast \ast) = \int_{\|\varrho_n\| \leq r_n} \left[ \rho(\theta_n^*)^T I_0^{-\frac{1}{2}} \varrho_n \exp \left( -\frac{n}{2} \varrho_n^T \varrho_n + O_P(g_r(\|\varrho_n\|)) \right) \right] d\varrho_n \\
\leq \int_{\|\varrho_n\| \leq r_n} \left[ \|\varrho_n\| \exp \left( -\frac{n}{2} \varrho_n^T \varrho_n \right) \right] d\varrho_n \\
+ \int_{\|\varrho_n\| \leq r_n} \left[ \|\varrho_n\| \exp \left( -\frac{n}{2} \varrho_n^T \varrho_n \right) \exp \left( O_P(g_r(\|\varrho_n\|)) \right) \right] d\varrho_n \\
\leq O_P(n^{-1}) + O_P(n^{-2r-1}/2) = O_P(n^{-1}),
\]
where $\theta_n^*$ is an intermediate value between $\hat{\theta}_n$ and $\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \varrho_n$. The last inequality follows from the analysis of $(\ast)$ when $1/4 < r < 1/2$. Hence we have proved that $(\ast \ast) = O_P(n^{-1})$ for $r > 1/4$. This completes the proof of lemma 3.2.□

Next we start the formal proof of theorem 3. Note first that
\[
\int_{\varrho_n \in \Xi_n} \left[ \rho(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \varrho_n) \frac{p_{l_n}(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \varrho_n)}{p_{l_n}(\hat{\theta}_n)} \right] d\varrho_n \\
= \int_{\{\|\varrho_n\| > r_n\} \cap \Xi_n} \left[ \rho(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \varrho_n) \frac{p_{l_n}(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \varrho_n)}{p_{l_n}(\hat{\theta}_n)} \right] d\varrho_n \\
+ \int_{\{\|\varrho_n\| \leq r_n\} \cap \Xi_n} \left[ \rho(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \varrho_n) \frac{p_{l_n}(\hat{\theta}_n + \tilde{I}_0^{-\frac{1}{2}} \varrho_n)}{p_{l_n}(\hat{\theta}_n)} \right] d\varrho_n.
\]
By lemma 3.3, the first integral on the right is of order $O_F(n^{-1/2}M_n(r))$. The second integral on the right can be decomposed into the following summands:

$$
\int_{\{||\theta_n|| \leq r_n\} \cap \Xi_n} \left[ \rho(\hat{\theta}_n + \hat{I}_0^{-1/2} \theta_n) pl_n(\hat{\theta}_n + \hat{I}_0^{-1/2} \theta_n) - \exp \left( -\frac{n}{2} \theta_n^T \theta_n \right) \rho(\theta_n) \right] d\theta_n
$$

$$
+ \int_{\{||\theta_n|| \leq r_n\} \cap \Xi_n} \left[ \exp \left( -\frac{n}{2} \theta_n^T \theta_n \right) \rho(\theta_n) \right] d\theta_n.
$$

The first part in the above is bounded by $O_F(n^{-1/2}M_n(r))$ via lemma 3.4. The second part equals

$$
n^{-1/2} \rho(\hat{\theta}_n) \int_{\{||\theta_n|| \leq \sqrt{n}r_n\} \cap \Xi_n} e^{-\frac{n}{2} \theta_n^T \theta_n} d\theta_n = n^{-1/2} \rho(\hat{\theta}_n) \int_{\mathbb{R}^d} e^{-\frac{n}{2} \theta_n^T \theta_n} d\theta_n + O(n^{-1/2}M_n(r)),
$$

where $u_n = \sqrt{n} \theta_n$. The above equality follows from the inequality $\int_{\mathbb{R}} e^{-y^2/2} dy \leq x^{-1} e^{-x^2/2}$ for any $x > 0$.

Consolidating the above analysis, we obtain

$$
(43) \int_{\theta_n \in \Xi_n} \left[ \rho(\hat{\theta}_n + \hat{I}_0^{-1/2} \theta_n) \frac{pl_n(\hat{\theta}_n + \hat{I}_0^{-1/2} \theta_n)}{pl_n(\hat{\theta}_n)} \right] d\theta_n = n^{-1/2} \rho(\hat{\theta}_n)(2\pi)^{d/2}
$$

$$
+ O_F(n^{-1/2}M_n(r)),
$$

and, by similar analysis, we also have

$$
(44) \int_{\theta_n \in (-\infty, n^{-1/2}\xi] \cap \Xi_n} \left[ \rho(\hat{\theta}_n + \hat{I}_0^{-1/2} \theta_n) \frac{pl_n(\hat{\theta}_n + \hat{I}_0^{-1/2} \theta_n)}{pl_n(\hat{\theta}_n)} \right] d\theta_n
$$

$$
= n^{-1/2} \rho(\hat{\theta}_n) \times \int_{(-\infty, \xi_1] \times \cdots \times (-\infty, \xi_d]} e^{-\frac{T_y}{2}} dy + O(n^{-1/2}M_n(r)).
$$

The quotient of (43) and (44) generates the desired error rate for fixed $\xi$. Note, however, that the above conclusions are unchanged if $\xi$ is replaced by an arbitrary sequence $\{\xi_n\} \in \mathbb{R}^d$. Thus (28) follows, proving theorem 3 in its entirety. \hfill \Box

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Proof of corollary 1. From the proof of theorem 3, we have
\[
\tilde{P}_\theta[\hat{\xi} \leq \xi] = \int_{\theta_n \in (-\infty, n^{-1/2}\xi]} \rho(\hat{\theta}_n + \bar{I}_0^{-\frac{1}{2}} \xi_n) \frac{p_{\theta_n}(\theta_n + \bar{I}_0^{-\frac{1}{2}} \xi_n)}{p_{\theta_n}(\theta_n)} d\xi_n.
\]
By differentiating both sides relative to \( \xi \) and combining with (43), we obtain
\[
f_n(\xi) = \frac{\rho \left( \hat{\theta}_n + \bar{I}_0^{-\frac{1}{2}} \xi \right)}{\left(2\pi\right)^{d/2} \rho(\hat{\theta}_n) + O_P(M_n(r))}.
\]
By analysis similar to the proof of corollary 2 in [5], the numerator equals \( \rho(\hat{\theta}_n) \exp(-\xi^T \xi/2) + O_P(M_n(r)) \). This completes the proof.\( \square \)

Proof of corollary 2. We only show (30) in what follows. Expression (31) can be shown similarly. The expansion in (30) is the quotient of two expansions of the form (43) and (44). We can show this as follows. First,
\[
\tilde{E}_\theta[\xi](\hat{\xi}) = \int_{\theta_n \in \Xi_n} \frac{\rho(\hat{\theta}_n + \bar{I}_0^{-\frac{1}{2}} \xi_n) p_{\theta_n}(\theta_n + \bar{I}_0^{-\frac{1}{2}} \xi_n)}{p_{\theta_n}(\theta_n)} d\xi_n.
\]
The denominator is \( n^{-1/2}(2\pi)^{d/2} \rho(\hat{\theta}_n) + O_P(n^{-1/2}M_n(r)) \) by (43). Similarly, by the proof of theorem 3 we know the numerator is a random vector of the order \( O_P(n^{-2r-1/2} + n^{-3/2}) \). This yields the desired conclusion.\( \square \)

Proof of theorem 4. By lemma 4.1 in [5], we can easily show that \( \kappa_n = \bar{I}_0^{-1/2} \xi + O_P(M_n(r)) \) for any \( \xi < \alpha < 1 - \xi \) and some choice of \( \zeta_\alpha \), where \( \xi \in (0, 1/2) \). Note that \( \kappa_n \) is not unique since the \( \alpha \)-th quantile of a \( d \) dimensional standard normal distribution, \( \zeta_\alpha \), is not unique when \( d > 1 \). The classical Edgeworth expansion implies that \( P(n^{-1/2} \sum_{i=1}^n \bar{I}_0^{-1/2} \xi_0(X_i) \leq \zeta_\alpha + a_n(\alpha)) = \alpha \), where \( a_n(\alpha) = O(n^{-1/2}) \), for \( \xi < \alpha < 1 - \xi \). This \( a_n(\alpha) \) is thus uniquely determined for each fixed \( \zeta_\alpha \) since \( \bar{I}_0(X_i) \) has at least one
absolutely continuous component. Let \( \hat{\kappa}_n = \sqrt{n}(\hat{\theta}_n - \theta_0) - \sum_{i=1}^{n} \tilde{I}_0(X_i) + \tilde{I}_0^{-1/2}a_n(\alpha) \). Then \( P(\sqrt{n}(\hat{\theta}_n - \theta_0) \leq \hat{\kappa}_n) = \alpha \). Combining with (21), we obtain \( \hat{\kappa}_n = \kappa_n + O_P(M_n(r)) \). The uniqueness of \( \hat{\kappa}_n \) follows from that of \( a_n(\alpha) \) for each fixed \( z_\alpha \), up to a term of order \( O_P(M_n(r)) \).

**Proof of theorem 5:** Under the assumptions of theorem 5, we next show that \( \chi^{n\alpha}_b = \chi^{2 \alpha}_d + O_P(M_n(r)) \) for \( \xi < \alpha < 1 - \xi \), where \( \xi \in (0, 1/2) \).

It is sufficient to show that \( \tilde{P}_{\theta|X}(PLR_b(\theta) \leq \chi^{2 \alpha}_d) = \alpha + O_P(M_n(r)) \) by considering the form of \( PLR_b(\theta) \) and (29). Based on the analysis in the proof of theorem 2, the term \( O_P(g_r(\|\hat{\theta}_n - \theta_n\|)) \) in (22) is actually bounded above by \( \Delta^{U}_{\text{cl}}(\hat{\theta}_n, \theta_n) \) and bounded below by \( \Delta^{L}_{\text{cl}}(\hat{\theta}_n, \theta_n) \). Thus it yields the inequality that \( n(\theta - \hat{\theta}_n)^T\tilde{I}_0(\theta - \hat{\theta}_n) - \Delta^{U}_{\text{cl}}(\theta, \hat{\theta}_n) \leq PLR_b(\theta) \leq n(\theta - \hat{\theta}_n)^T\tilde{I}_0(\theta - \hat{\theta}_n) - \Delta^{L}_{\text{cl}}(\theta, \hat{\theta}_n) \) such that we have constructed the upper bound and lower bound for \( \tilde{P}_{\theta|X}(PLR_b(\theta) \leq \chi^{2 \alpha}_d) \).

We next show that the upper and lower bound matches asymptotically at \( \alpha + O_P(M_n(r)) \). Without loss of generality, we only consider its upper bound in what follows.

\[
\tilde{P}_{\theta|X}(n(\theta - \hat{\theta}_n)^T\tilde{I}_0(\theta - \hat{\theta}_n) \leq \chi^{2 \alpha}_d + \Delta^{U}_{\text{cl}}(\theta, \hat{\theta}_n))
\]

\[
\leq \tilde{P}_{\theta|X}(W_n) + \tilde{P}_{\theta|X}(\|\varepsilon_n\| > r_n)
\]

\[
\leq \tilde{P}_{\theta|X}(W_n) + \int_{\{\|\varepsilon_n\| > r_n\} \cap \Xi_n} \rho(\hat{\theta}_n + \frac{\tilde{I}_0^{-\alpha}}{2} \varepsilon_n) \rho_n(\hat{\theta}_n + \frac{\tilde{I}_0^{-\alpha}}{2} \varepsilon_n) d\varepsilon_n
\]

\[
\leq \tilde{P}_{\theta|X}(W_n) + O_P(n^{-M}),
\]

where \( r_n = o(n^{1/3}) \) with \( \sqrt{nr_n} \to \infty \), \( W_n = \{n\varepsilon_n^T\varepsilon_n \leq \chi^{2 \alpha}_d + \Delta^{U}_{\text{cl}}(\theta, \hat{\theta}_n)\} \cap \{\|\varepsilon_n\| \leq r_n\} \), and \( M \) is an arbitrary positive number. The third inequality

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above follows from lemma 3.1 and (43).

We next study \( \tilde{P}_{\theta_1, \tilde{X}}(W_n) \). Accordingly,

\[
\tilde{P}_{\theta_1, \tilde{X}}(W_n) = \frac{\int_{W_n} \rho(\hat{\theta}_n + \frac{1}{n} \epsilon_n) \frac{p_{n}(\hat{\theta}_n + \frac{1}{n} \epsilon_n)}{p_n(\hat{\theta}_n)} d\epsilon_n}{\int_{\mathbb{R}^d} \rho(\hat{\theta}_n + \frac{1}{n} \epsilon_n) \frac{p_{n}(\hat{\theta}_n + \frac{1}{n} \epsilon_n)}{p_n(\hat{\theta}_n)} d\epsilon_n}.
\]

\[
= \int_{W_n} \left[ \rho(\hat{\theta}_n + \frac{1}{n} \epsilon_n) \frac{p_{n}(\hat{\theta}_n + \frac{1}{n} \epsilon_n)}{p_n(\hat{\theta}_n)} - \rho(\hat{\theta}_n) \exp\left( -\frac{1}{2} \epsilon_n^T \epsilon_n \right) \right] d\epsilon_n
\]

\[
= \frac{n^{-1/2} \rho(\hat{\theta}_n)(2\pi)^{d/2} + O_P(n^{-1/2} M_n(r))}{n^{-1/2} \rho(\hat{\theta}_n)(2\pi)^{d/2} + O_P(n^{-1/2} M_n(r))}
\]

\[
= O_P(M_n(r)) + \frac{\int_{W_n} \rho(\hat{\theta}_n) \exp\left( -\frac{1}{2} \epsilon_n^T \epsilon_n \right) d\epsilon_n}{n^{-1/2} \rho(\hat{\theta}_n)(2\pi)^{d/2} + O_P(n^{-1/2} M_n(r))}
\]

\[
= \int_{V_n} \rho(\hat{\theta}_n) \exp\left( -\frac{1}{2} \epsilon_n^T \epsilon_n \right) d\epsilon_n + \int_{W_n - V_n} \rho(\hat{\theta}_n) \exp\left( -\frac{1}{2} \epsilon_n^T \epsilon_n \right) d\epsilon_n
\]

\[
= \alpha + \frac{\int_{W_n - V_n} \rho(\hat{\theta}_n) \exp\left( -\frac{1}{2} \epsilon_n^T \epsilon_n \right) d\epsilon_n}{n^{-1/2} \rho(\hat{\theta}_n)(2\pi)^{d/2} + O_P(n^{-1/2} M_n(r))} + O_P(M_n(r)),
\]

where \( V_n = \{ n \epsilon_n^T \epsilon_n \leq \chi^2_{d, \alpha} \} \). The third equality in the above follows from (43) and lemma 3.2 in the proof of theorem 3. We next study the fraction in the last equality above. It is easy to show that \( \{ W_n - V_n \} \subseteq \{ W_n - V_n \cap \{ \| \epsilon_n \| \leq r_n \} \} \subseteq \{ \chi^2_{d, \alpha} \leq n \epsilon_n^T \epsilon_n \leq \chi^2_{d, \alpha} + \Delta_{\epsilon_n}^U(\theta, \hat{\theta}_n) \} \cap \{ \| \epsilon_n \| \leq r_n \} \equiv T_n \). By replacing \( \sqrt{n} \epsilon_n \) with \( u_n \), \( T_n \) can be reexpressed as \( \{ \chi^2_{d, \alpha} \leq u_n^T u_n \leq \chi^2_{d, \alpha} + \Delta_{\epsilon_n}^U(\theta, \hat{\theta}_n) \} \cap \{ \| u_n \| \leq \sqrt{r_n} \} \).

We next consider the order of \( \int_{W_n - V_n} \rho(\hat{\theta}_n) \exp\left( -\frac{1}{2} \epsilon_n^T \epsilon_n / 2 \right) d\epsilon_n \) for \( r \) in different ranges. For \( r \geq 1/2 \), \( \Delta_{\epsilon_n}^U(\theta, \hat{\theta}_n) = O_P(n^{-1/2} + n^{-1/2} \| u_n \|^3) \). Under the condition that \( \| u_n \| \leq \sqrt{r_n} \), \( \Delta_{\epsilon_n}^U(\theta, \hat{\theta}_n) = o_P(1) \). Hence any subsequence of \( u_n \) contained in \( T_n \) is not diverging. In this case, \( \Delta_{\epsilon_n}^U(\theta, \hat{\theta}_n) = O_P(n^{-1/2}). \)
In summary we have the following inequalities:

\[
\int_{W_n - V_n} \rho(\hat{\theta}_n) \exp \left( -\frac{n}{2} \hat{\theta}_n^T \hat{\theta}_n \right) \, d\theta_n \leq \int_{T_n} \rho(\hat{\theta}_n) \exp \left( -\frac{n}{2} \hat{\theta}_n^T \hat{\theta}_n \right) \, d\theta_n \\
\leq n^{-1/2} \int_{Q_n} \rho(\hat{\theta}_n) \exp \left( -\frac{u_n^T u_n}{2} \right) \, du_n \\
\leq O_P(n^{-1}),
\]

where \( Q_n = \{ \chi_{d,\alpha}^2 \leq u_n^T u_n \leq \chi_{d,\alpha}^2 + O_P(n^{-1/2}) \} \cap \{ u_n \leq \sqrt{n} r \}. \) Hence \( \hat{P}_{\theta|X}(W_n) = \alpha + O_P(n^{-1/2}) \) when \( r \geq 1/2. \)

Similar arguments will now be applied to the case \( 1/4 < r < 1/2. \) It is sufficient to show that \( \int_{W_n - V_n} \rho(\hat{\theta}_n) \exp(-\frac{n}{2} \hat{\theta}_n^T \hat{\theta}_n) \, d\theta_n = O_P(n^{-2r}) \) for \( r \) in the above range. When \( 1/3 \leq r < 1/2 \) and \( \| u_n \| \leq \sqrt{n} \), \( \Delta_{5n}^U(\theta, \hat{\theta}_n) \) converges to zero in probability since \( \Delta_{5n}^U(\theta, \hat{\theta}_n) = O_P(n^{-1/2} \| u_n \|^3 + n^{-r} \| u_n \|^2 + n^{1/2-2r} \| u_n \| + n^{-2r+1/2}). \) Consequently, \( \Delta_{5n}^U(\theta, \hat{\theta}_n) \) is \( O_P(n^{1/2-2r}) \) by the analysis we used for the case when \( r \geq 1/2. \) However, for \( 1/4 < r < 1/3, \) \( \Delta_{5n}^U(\theta, \hat{\theta}_n) = O_P(n^{-2r+1/2} \| u_n \| + n^{-2r+1/2}). \) By making the same choice of \( r_n \) used in the proof of lemma 3.2, we have \( \Delta_{5n}^U(\theta, \hat{\theta}_n) = o_P(1). \) Hence there does not exist a diverging subsequence of \( u_n \) contained in \( T_n \) when we choose this specific \( r_n. \) This implies that \( \Delta_{5n}^U(\theta, \hat{\theta}_n) = O_P(n^{1/2-2r}). \) In other words, \( \Delta_{5n}^U(\theta, \hat{\theta}_n) = O_P(n^{1/2-2r}) \) for \( 1/4 < r < 1/2. \) This implies that \( \int_{W_n - V_n} \rho(\hat{\theta}_n) \exp(-\frac{n}{2} \hat{\theta}_n^T \hat{\theta}_n) \, d\theta_n = O_P(n^{-2r}). \) The same arguments also apply to the lower bound of \( \hat{P}_{\theta|X}(PLR_b(\theta) \leq \chi_{d,\alpha}^2). \) Thus we have shown that \( \chi_{b,\alpha}^2 = \chi_{d,\alpha}^2 + O_P(M_n(r)) \) for \( \xi < \alpha < 1 - \xi, \) where \( \xi \in (0,1/2). \)

If we can show that \( \chi_{b,\alpha}^2 = \chi_{d,\alpha}^2 + O_P(M_n(r)) \), then the whole proof is complete. Combining (21) and (22) in theorem 2, we can rewrite \( PLR_f(\theta_0) \) as \( n^{-1} \sum_{i=1}^{n} \tilde{f}_0(X_i)^T \tilde{f}_0^{-1} \sum_{i=1}^{n} \tilde{f}_0(X_i) + O_P(M_n(r)). \) By classical Edgeworth expansion, we have \( P(n^{-1/2} \sum_{i=1}^{n} \tilde{f}_0^{-1/2} \tilde{f}_0(X_i) \leq z_{\alpha}) = \alpha + O(n^{-1/2}), \) which
directly yields \( P(n^{-1} \sum_{i=1}^{n} \tilde{\ell}_0(X_i)^T \tilde{J}_0^{-1} \sum_{i=1}^{n} \tilde{\ell}_0(X_i)^T \leq \chi^2_{d,a} + O(n^{-1/2})) = \alpha \). Thus \( \chi^2_f = \chi^2_{d,a} + O_P(M_n(r)) \). This completes the proof. \( \square \)

Proof of lemma 2: We first review some known results from [16] about the Cox model with current status data. For some constant \( C \) and every \( x \) (under the above regularity conditions), we have

\[
|\text{lik}(\theta_0, \Lambda_0)(x) - \text{lik}(\theta_0, \Lambda)(x)| \leq C|\Lambda(y) - \Lambda_0(y)|, \\
|\hat{\ell}(\theta_0, \theta_0, \Lambda)(x) - \hat{\ell}(\theta_0, \theta_0, \Lambda_0)(x)| \leq C|\Lambda(y) - \Lambda_0(y)|, \\
|\text{lik}(\theta_0, \Lambda)(x) - \text{lik}(\theta_0, \Lambda_0)(x) - A_0(\Lambda - \Lambda_0)(x)\text{lik}(\theta_0, \Lambda_0)(x)| \\
\leq |\Lambda(y) - \Lambda_0(y)|^2,
\]

where \( A_0 = A_{\theta_0, \Lambda_0} \) and \( A_{\theta, \Lambda} \) is the score operator for \( \Lambda \) at \((\theta, \Lambda)\), e.g., the Fréchet derivative of \( \log p_{\theta, \Lambda} \) relative to \( \Lambda \). Thus by the decomposition of \( P\hat{\ell}(\theta_0, \theta_0, \Lambda) \) in what follows, we can show (16) holds with the \( L_2 \) norm on \( \Lambda \):

\[
P\hat{\ell}(\theta_0, \theta_0, \Lambda) = \\
P\left[ \frac{p_0 - p_{\theta_0, \Lambda}}{p_0} (\hat{\ell}(\theta_0, \theta_0, \Lambda) - \hat{\ell}_0) \right] - P\hat{\ell}(\theta_0, \theta_0, \Lambda_0) \left[ \frac{p_{\theta_0, \Lambda} - p_0}{p_0} - A_0(\Lambda - \Lambda_0) \right].
\]

Next we can show (14) by the following inequality:

\[
P(\hat{\ell}(\theta_0, \theta_0, \Lambda) - \hat{\ell}(\theta_0, \theta_0, \Lambda_0)) \\
\leq P |N(\theta_0, \theta_0, \Lambda) - N(\theta_0, \theta_0, \Lambda_0)| + P \left| \hat{\ell}^2(\theta_0, \theta_0, \Lambda) - \hat{\ell}^2(\theta_0, \theta_0, \Lambda_0) \right| \\
\leq P |N(\theta_0, \theta_0, \Lambda) - N(\theta_0, \theta_0, \Lambda_0)| + C\|\Lambda - \Lambda_0\|_{L_2},
\]

where \( N(t, \theta, \Lambda) = (\partial^2 \text{lik}(t, \Lambda_4(\theta, \Lambda))/\partial \lambda^2)/\text{lik}(t, \Lambda_4(\theta, \Lambda)) \). The second inequality follows from the boundedness of \( \hat{\ell}(t, \theta, \Lambda) \) and (45). We next proceed.
to derive an upper bound for $P \mid N(\theta_0, \theta_0, \Lambda) - N(\theta_0, \theta_0, \Lambda_0)\mid$:

\[
P \mid N(\theta_0, \theta_0, \Lambda) - N(\theta_0, \theta_0, \Lambda_0)\mid \\
\lesssim P \mid \Lambda Q_x(\theta_0, \Lambda) - \Lambda Q_x(\theta_0, \Lambda_0)\mid + P \mid \Lambda - \Lambda_0\mid + P \mid w(\Lambda) - w(\Lambda_0)\mid \\
+ P \mid w(\Lambda)\Lambda - w(\Lambda_0)\Lambda_0\mid \\
\lesssim P \mid \Lambda Q_x(\theta_0, \Lambda) - \Lambda Q_x(\theta_0, \Lambda_0)\mid + \|\Lambda - \Lambda_0\|_{L^2} \\
\lesssim \|\Lambda - \Lambda_0\|_{L^2},
\]

where $w(\Lambda) = \phi(\Lambda) h_{00} \circ \Lambda_0^{-1} \circ \Lambda$. Clearly, $w(\Lambda_0) = h_0$. Note that $w(\Lambda)$ can be expressed as $\Lambda \varsigma(\Lambda) u(\Lambda)$, where $\varsigma(\Lambda) = \phi(\Lambda)/\Lambda$ and $u(\Lambda) = h_{00} \circ \Lambda_0^{-1} \circ \Lambda$. Note that $\varsigma(\Lambda)$ and $u(\Lambda)$ are both assumed bounded and Lipschitz. Hence $P \mid w(\Lambda) - w(\Lambda_0)\mid \lesssim \|\Lambda - \Lambda_0\|_{L^2}$ and $P \mid \Lambda w(\Lambda) - \Lambda_0 w(\Lambda_0)\mid \lesssim \|\Lambda - \Lambda_0\|_{L^2}$. This explains the second inequality in the above. The inequality that $P \mid \Lambda Q_x(\theta_0, \Lambda) - \Lambda Q_x(\theta_0, \Lambda_0)\mid \lesssim \|\Lambda - \Lambda_0\|_{L^2}$ follows from the inequality that $|(u(e^u - e^v))/(1(e^u - 1)(e^v - 1))| \lesssim |u - v|$ given that $u \geq 0$ in some compact set and $v > 0$ in some compact set. Combining this with the previous analysis, we can conclude that (14) holds under the given assumptions for the Cox model with current status data. Similar techniques can be applied to the verification of (15). We omit the details.

Finally, we only need to check (13). Note that $\mathbb{G}_n(\hat{\ell}(\theta_0, \theta_0, \hat{\Lambda}_\theta) - \hat{\ell}_0)$ can be written as follows:

\[
\mathbb{G}_n \left( (\hat{\Lambda}_\theta - \Lambda_0)zQ(x; \theta_0, \Lambda_0) \right) - \mathbb{G}_n \left( (w(\hat{\Lambda}_\theta) - w(\Lambda_0))Q(x; \theta_0, \Lambda_0) \right) \\
+ \mathbb{G}_n \left( Q(x; \theta_0, \hat{\Lambda}_\theta)(z\hat{\Lambda}_\theta - w(\hat{\Lambda}_\theta)) - Q(x; \theta_0, \Lambda_0)(z\Lambda_0 - w(\Lambda_0)) \right) \\
- \mathbb{G}_n \left( (z\hat{\Lambda}_\theta - w(\hat{\Lambda}_\theta)) - (z\Lambda_0 - w(\Lambda_0)) \right) Q(x; \theta_0, \Lambda_0).
\]

To verify (13), we need to make use of the following technical tools:
T1. (Lemma 5.13 in [23], page 79) Consider a uniformly bounded class of functions $\mathcal{G}$, with $\sup_{g \in \mathcal{G}} \|g - g_0\|_\infty < \infty$ and $\log N_F(\epsilon, \mathcal{G}, P) \leq A \epsilon^{-\alpha}$ for all $\epsilon > 0$, and where $\alpha \in (0, 2)$. Then for $\delta_n = n^{-(2+\alpha)/(2+\alpha)}$, 

$$ \sup_{g \in \mathcal{G}} \frac{|(\mathbb{P}_n - P)(g - g_0)|}{\|g - g_0\|_1^{1-\alpha/2} \sqrt{n \delta_n^2}} = O_P(n^{-1/2}). $$

T2. (Theorem 2.7.11 in [24]) Let $\mathcal{F} = \{f_t : t \in T\}$ be a class of functions satisfying $|f_s(x) - f_t(x)| \leq d(s, t)F(x)$ for every $s$ and $t$ and some fixed function $F$. Then, for any norm $\| \cdot \|$, $N_F(2\epsilon\|F\|, \mathcal{F}, \| \cdot \|) \leq N(\epsilon, T, d)$.

Clearly, by T1 we have $\mathbb{G}_n \left((\hat{\lambda}_{\hat{\theta}_n} - \Lambda_0)zQ(x; \theta_0, \Lambda_0)\right) = O_P(n^{-1/6} + \|\hat{\lambda}_{\hat{\theta}_n} - \Lambda_0\|_{L_2}^{1/2})$, since $\alpha = 1$ for monotone functions $\Lambda$. Then by the relation $O_P(\|\hat{\lambda}_{\hat{\theta}_n} - \Lambda_0\|_{L_2}^{1/2}) = O_P((\|\hat{\theta}_n - \theta_0\| + n^{-1/3})^{1/2}) = O_P(n^{-1/6} + n^{1/6}|\hat{\theta}_n - \theta_0|)$, we know that the first line of (46) satisfies (13). Note that the class of Lipschitz functions of $\Lambda$ also has the same upper bound (i.e., $\alpha = 1$) for the entropy with bracketing number by T2 and the inequality that $N(\epsilon, \mathcal{G}, \| \cdot \|_{L_2}) \leq N(2\epsilon, \mathcal{G}, \| \cdot \|_{L_2})$. This now implies that $\mathbb{G}_n \left((w(\hat{\lambda}_{\hat{\theta}_n}) - w(\Lambda_0))Q(x; \theta_0, \Lambda_0)\right) = O_P(n^{-1/6} + n^{1/6}|\hat{\theta}_n - \theta_0|)$ since $w(\Lambda)$ is Lipschitz in $\Lambda$. Similar arguments apply to the other lines in (46). □

Proof of lemma 3: The proof is analogous to that of lemma 2 in [12]. □

Proof of lemma 4: We apply theorem 1 with $m_{\theta, k} = \lambda(\theta, k)$, where $\lambda(\theta, k) \equiv \log \text{lik}(\theta, k)$, since $\text{lik}(\theta, k)$ is bounded away from zero and infinity for $(\theta, k) \in \Theta \times \mathcal{O}_2^M$. It suffices to show (17) provided both $P(\lambda(\theta, k_0) - \lambda(\theta_0, k_0)) \approx -\|\theta - \theta_0\|^2$ and $P(\lambda(\theta, k) - \lambda(\theta_0, k_0)) \approx -d_{\theta}^2(k, k_0)$ hold. Note that the maximality of the point $(\theta_0, k_0)$ around the criterion function $(\theta, k) \mapsto P\lambda(\theta, k)$ implies that $P(\lambda(\theta, k_0) - \lambda(\theta_0, k_0)) \approx -\|\theta - \theta_0\|^2$. By using the inequality $P\log(q/p) \leq -h^2(p, q)$ and the relationship between Kullback-Leibler divergence and squared Hellinger distance, we can show that $P(\lambda(\theta, k) -$
\[ \lambda(\theta_0, k_0) \leq -f(\sqrt{p_{\theta,k}} - \sqrt{p_0})^2 \, d\mu \leq -\|p_{\theta,k} - p_0\|_{L_2}^2. \] Hence \( d_\theta(k, k_0) = \|p_{\theta,k} - p_0\|_{L_2} \). Thus we only need to verify condition (18) by lemma 1 to complete the whole proof.

Condition (20) in lemma 1 trivially holds by considering the forms of \( m_{\theta,k} \) and \( d_\theta(k, k_0) \). By theorem 1, we can show that \( d_{\hat{\theta}_n}(\hat{k}_{\hat{\theta}_n}, k_0) = O_P(\delta_n + \|\hat{\theta}_n - \theta_0\|) \) for any \( \delta_n \) satisfying \( K(\delta_n, S_{\hat{\theta}_n}(L_2(P))) \leq \sqrt{n} \delta_n^2 \), where the function \( K \) is defined in (19). In other words, we need to calculate the \( \epsilon \)-bracketing entropy number for the class of functions \( S_{\hat{\theta}_n} \). To achieve the desired rate (33), we only need to show \( H_B(\epsilon, S_{\hat{\theta}_n}, L_2(P)) \leq \epsilon^{-1/2} \) based on the above discussions. Recall that \( S_{\hat{\theta}_n} = \{ x \mapsto \lambda(\theta, k)(x) - \lambda(\theta, k_0)(x) : d_\theta(k, k_0) < \delta_n, \|\theta - \theta_0\| < \delta_n \} \). By considering technical tool T3 below, we only need to show that \( H_B(\epsilon, C, L_2(P)) \leq \epsilon^{-1/2} \), where \( C = \{ x \mapsto \lambda(\theta, k)(x) : J_2(k) + \|k - k_0\|_\infty \leq C_1, \|\theta - \theta_0\| \leq C_1 \} \).

T3. (Lemma 9.24 in [9]) Let \( \mathcal{F} \) and \( \mathcal{G} \) be classes of measurable functions. Then for any probability measure \( Q \) and any \( 1 \leq r \leq \infty \),

\[ H_B(2\epsilon, \mathcal{F} + \mathcal{G}, L_r(Q)) \leq H_B(\epsilon, \mathcal{F}, L_r(Q)) + H_B(\epsilon, \mathcal{G}, L_r(Q)). \]

Now we consider \( C_1 = \{ q_{\theta,k}(x)/(1 + J(k)) : \|k - k_0\|_\infty \leq C_1, \|\theta - \theta_0\| \leq C_1 \} \).

By technical tool T4 below, we obtain \( H_B(\epsilon, C_1, L_2(P)) \leq \epsilon^{-1/2} \) as desired.

T4. (See [4]) For each \( 0 < C < \infty \) and \( \delta > 0 \) we have

\[ H_B(\delta, \{ \eta : \|\eta\|_\infty \leq C, J_k(\eta) \leq C \}, \|\cdot\|_\infty) \geq \left( \frac{C}{\delta} \right)^{1/k}. \]

Continuing, note that \( \lambda(\theta, k)(X) \) can be rewritten as:

\[ \Delta \log \Phi(q_{\hat{\theta},k}A) + (1 - \Delta) \log (1 - \Phi(q_{\hat{\theta},k}A)), \]

where \( A = 1 + J(k) \) and \( q_{\hat{\theta},k} \in C_1 \). We next calculate the \( \epsilon \)-bracketing entropy number with the \( L_2 \) norm for the class of functions \( R_1 = \{ k_a(t) : \).
$t \mapsto \log \Phi(at)$ for $a \geq 1$ and $t \in T$, where $T$ is some bounded subset in $\mathbb{R}^d$. Note that $k_a(t)$ is increasing (decreasing) in $a$ for $t > 0$ ($t < 0$). After some derivation, we obtain that $\sup_{t \in T} |k_a(t) - k_b(t)| \lesssim |a - b|$ for any fixed $a, b > 1$ and $\sup_{a, b \geq A_0, t \in T} |k_a(t) - k_b(t)| \lesssim A_0^{-1}$. The above two inequalities imply that the $\epsilon$-bracketing number with uniform norm is of order $O(\epsilon^{-2})$ for $a \in [1, \epsilon^{-1}]$ and is 1 for $a > \epsilon^{-1}$. Thus we know $H_B(\epsilon, R_1, L_2) = O(\log \epsilon^{-2})$. By applying a similar analysis to $R_2 \equiv \{k_a(t) : t \mapsto \log(1 - \Phi(at))$ for $a \geq 1$ and $t \in T\}$, we obtain that $H_B(\epsilon, R_2, L_2) = O(\log \epsilon^{-2})$. This combined with technical tool T5 below, yields that $H_B(\epsilon, C, L_2) \lesssim \epsilon^{-1/2}$. Thus far we have shown that $d_{\theta_n}(\hat{k}_{\theta_n}, \theta_0) = O_P(n^{-2/5} + \|\hat{\theta}_n - \theta_0\|)$. Now by considering the usual Taylor expansion and the assumption that $EVaR(W|Z)$ is positive definite, we have verified (33). □

T5. (Lemma 15.2 in [9]) For a probability measure $P$, let $\mathcal{F}_1$ be a class of measurable functions $f_1 : X \mapsto \mathbb{R}$, and let $\mathcal{F}_2$ denote a class of nondecreasing functions $f_2 : \mathbb{R} \mapsto [0, 1]$ that are measurable for every probability measure. Then,

$$H_B(\epsilon, \mathcal{F}_2(\mathcal{F}_1), L_2(P)) \leq 2H_B(\epsilon/3, \mathcal{F}_1, L_2(P)) + \sup_Q H_B(\epsilon/3, \mathcal{F}_2, L_2(Q)).$$

Proof of lemma 5. Note that $k \in O_2^M$. Hence we can easily verify assumption B1 since every map $(t, \theta, k) \mapsto (\partial^l+m/\partial t^l \partial \theta^m)\ell(t, \theta, k)$ is uniformly bounded. Note that $(C, W)$ lies in some bounded set and $h_0(\cdot)$ is bounded. Hence we can show that the Fréchet derivatives of $k \mapsto \tilde{\ell}(\theta_0, \theta, k)$ and $k \mapsto \ell_t(\theta_0, \theta_0, k)$ for any $k \in O_2^M$ are bounded operators, i.e., $|\tilde{\ell}(\theta_0, \theta_0, k)(X) - \ell_0(X)|$ is bounded by the product of some integrable function and $|k - k_0|(Z)$. Thus (14) and (15) are satisfied, and the bounded Fréchet derivative of $k \mapsto \ell(\theta_0, \theta_0, k)$ plus second order Fréchet differentiability of $k \mapsto lik(\theta_0, k)$.
implies (16).

Since the convergence rate \( r = 2/5 \), it suffices to show the asymptotic equicontinuity condition (13), provided (50) holds. Accordingly,

\[
G_n (\hat{\ell} (\theta_0, \theta_0, \hat{k}_{\theta_0}) - \hat{\ell}_0) = O_P ((n^{-3/10} + n^{1/10}) \| \tilde{\theta}_n - \theta_0 \|).
\]

To show (50), we need the following technical tool T6:

T6. (Lemma 3.4.2 in [24]) Let \( \mathcal{F} \) be a class of measurable functions such that \( Pf^2 < \delta^2 \) and \( \| f \|_\infty \leq M \) for every \( f \) in \( \mathcal{F} \). Then

\[
E_P \| G_n \|_{\mathcal{F}} \lesssim K(\delta, \mathcal{F}, L_2(P)) \left( 1 + \frac{K(\delta, \mathcal{F}, L_2(P))}{\delta^2 n^{1/2}} M \right),
\]

where \( K(\delta, \mathcal{F}, \| \cdot \|) = \int_0^\delta \sqrt{1 + H_B(\epsilon, \mathcal{F}, \| \cdot \|)} \, d\epsilon. \)

To utilize this tool, note first that (33) implies:

\[
P \left( \frac{\hat{\ell} (\theta_0, \theta_0, \hat{k}_{\theta_0}) - \hat{\ell}_0}{n^{-3/10} + n^{1/10} \| \tilde{\theta}_n - \theta_0 \|} \right)^2 \lesssim O_P \left( n^{-\frac{2}{5}} \right).
\]

We next define the set \( Q_n \) as \( \left\{ g \in L_2(P) : P g^2 \leq C_n n^{-\frac{2}{5}} \right\} \)

\[
\cap \left\{ \frac{\hat{\ell} (\theta_0, \theta_0, k) - \hat{\ell}_0}{n^{-3/10} + n^{1/10} \| \theta - \theta_0 \|} : k \in O_2^M, \| \theta - \theta_0 \| \leq \delta \right\},
\]

for some \( \delta > 0 \). Obviously the function \( (\hat{\ell} (\theta_0, \theta_0, \hat{k}_{\theta_0}) - \hat{\ell}_0)/(n^{-3/10} + \tilde{\theta}_n - \theta_0 \|) \) \( \in Q_n \) on a set of probability arbitrarily close to one, as \( C_n \to \infty \). If we can show \( \lim_{n \to \infty} E^* \| G_n \|_{Q_n} < \infty \) by T6, then the proof of lemma 5 is complete.

Note that \( \hat{\ell} (\theta_0, \theta_0, k) \) depends on \( k \) in a Lipschitz manner. Consequently we can bound \( H_B (\epsilon, Q_n, L_2(P)) \) by the product of some constant and \( H (\epsilon, R_n, L_2(P)) \), where \( R_n \) is defined as \( \{ G_n (k) : J (G_n (k)) \leq n^{3/10}, \| G_n (k) \|_{\infty} \leq n^{3/10} \} \), and where \( G_n (k) = k/(n^{-3/10} + n^{1/10} \| \theta - \theta_0 \|) \). By the main results in [4], we know
$H(\epsilon, \mathcal{R}_n, L_2(P)) \lesssim (n^{3/10}/\epsilon)^{1/k}$. Note that $\delta_n = n^{-1/10}$ and $M_n = n^{3/10}$ in T6. Thus by calculation using T6, we can establish that $\lim_{n \to \infty} E^{*} \| G_n \| Q_n < \infty. \square$

**Proof of lemma 6:** By the assumption that $\Delta_n(\tilde{\theta}_n) = o_P(1)$, we have $\Delta_n(\tilde{\theta}_n) - \Delta_n(\theta_0) \geq o_P(1)$. Thus the following inequality holds:

$$n^{-1} \sum_{i=1}^{n} \log \left( \frac{H(\tilde{\theta}_n, \hat{k}_{\theta_0}; X_i)}{H(\theta_0, \hat{k}_{\theta_0}; X_i)} \right) \geq o_P(1),$$

where $H(\theta, k; X) = \Delta \Phi(C - \theta W - k(Z)) + (1 - \Delta)(1 - \Phi(C - \theta W - k(Z)))$.

By the assumptions on $k$, we know that $H(\tilde{\theta}_n, \hat{k}_{\theta_0}; X_i)$ belongs to some $P$-Donsker class. Combining the above conclusion and the inequality $\alpha \log x \leq \log(1 + \alpha \{x - 1\})$ for some $\alpha \in (0, 1)$ and any $x > 0$, we can show that

$$P \log \left[ 1 + \alpha \left( \frac{H(\tilde{\theta}_n, \hat{k}_{\theta_0}; X_i)}{H(\theta_0, \hat{k}_{\theta_0}; X_i)} - 1 \right) \right] \geq o_P(1)$$

(51)

The strict concavity of $x \mapsto \log(1 + \alpha(x - 1))$ ensures that

$$P \log \left[ 1 + \alpha \left( \frac{H(\tilde{\theta}_n, \hat{k}_{\theta_0}; X_i)}{H(\theta_0, \hat{k}_{\theta_0}; X_i)} - 1 \right) \right] \leq 0$$

This combined with (51) implies that

$$P \log \left[ 1 + \alpha \left( \frac{H(\tilde{\theta}_n, \hat{k}_{\theta_0}; X_i)}{H(\theta_0, \hat{k}_{\theta_0}; X_i)} - 1 \right) \right] = o_P(1)$$

The strict concavity of $x \mapsto \log(1 + \alpha(x - 1))$ forces the result that $P|\Phi(C - \tilde{\theta}_n W - \hat{k}_{\theta_0}(Z)) - \Phi(C - \theta_0 W - \hat{k}_{\theta_0}(Z))| = o_P(1)$. The desired conclusion now follows from model identifiability. \square

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