Lecture 5 - Discrete Distributions

Sta102 / BME102

Colin Rundel

September 8, 2014

Random variables

- A *random variable* is a numeric quantity whose value depends on the outcome of a random event
  - We use a capital letter, like X, to denote a random variables
  - The values of a random variable will be denoted with a lower case letter, in this case x
  - For example, $P(X = x)$
- There are two types of random variables:
  - *Discrete random variables* take on only integer values
    - Example: Number of credit hours, Difference in number of credit hours this term vs last
  - *Continuous random variables* take on real (decimal) values
    - Example: Cost of books this term, Difference in cost of books this term vs last

Discrete Probability distributions

A *discrete probability distribution* lists all possible events and the probabilities with which they occur.

- The probability distribution for the gender of one child:

Event	B	G
Probability	0.5	0.5

- Rules for probability distributions:
  1. The events listed must be disjoint
  2. Each probability must be between 0 and 1
  3. The probabilities must add up to 1

Example - Discrete probability model

In a game of cards you win \$1 if you draw a heart, \$5 if you draw an ace (including the ace of hearts), \$10 if you draw the king of spades and nothing for any other card you draw. Write the probability distribution for the random variable representing your winnings.

Mean and standard deviation of a discrete RVs

We are often interested in the value we expect to arise from a random variable.

- We call this the expected value, it is a weighted average of the possible outcomes

$$E(X) = \sum_x x \cdot P(X = x)$$

We are also often interested in the variability in the values of a random variable.

- Described using Variance and Standard deviation

$$\begin{aligned} Var(X) &= E[(X - E(X))^2] \\ &= \sum_x (x - E(X))^2 \cdot P(X = x) \\ SD(X) &= \sqrt{Var(X)} \end{aligned}$$

Example - Discrete RV - Mean and SD

For the previous example what is the expected value and the standard deviation of your winnings.

X	$P(X)$	$X \cdot P(X)$	$(X - E(X))^2$	$P(X) \cdot (X - E(X))^2$
0	$\frac{35}{52}$	$0 \times \frac{35}{52} = 0$	$(0 - 0.81)^2 = 0.6561$	$\frac{35}{52} \times 0.6561 = 0.4416$
1	$\frac{12}{52}$	$1 \times \frac{12}{52} = \frac{12}{52}$	$(1 - 0.81)^2 = 0.0361$	$\frac{12}{52} \times 0.0361 = 0.0083$
5	$\frac{4}{52}$	$5 \times \frac{4}{52} = \frac{20}{52}$	$(5 - 0.81)^2 = 17.5561$	$\frac{4}{52} \times 17.5561 = 1.3505$
10	$\frac{1}{52}$	$10 \times \frac{1}{52} = \frac{10}{52}$	$(10 - 0.81)^2 = 84.4561$	$\frac{1}{52} \times 84.0889 = 1.6242$

Bernoulli Random Variable

A Bernoulli random variable describes a trial with only two possible outcomes, one of which we will label a success and the other a failure and where the probability of a success is given by the parameter p. (Since it needs to be numeric) the random variable takes the value 1 to indicate a success and 0 to indicate a failure.

X	P(X=x)
0	1-p
1	p

$$P(X = x|p) = \begin{cases} p & \text{if x=1} \\ 1 - p & \text{if x=0} \end{cases}$$

Properties of a Bernoulli Random Variable

Let $X \sim \text{Bern}(p)$ then

$$\begin{aligned} E(X) &= \sum_x x\ P(X = x) \\ &= 0 \cdot P(X = 0) + 1 \cdot P(X = 1) \\ &= P(X = 1) \\ &= p \end{aligned}$$

$$\begin{aligned} Var(X) &= E(X - p)^2 \\ &= E(X^2 - 2Xp + p^2) \\ &= E(X^2) - 2p\ E(X) + p^2 \\ &= (0^2 \cdot P(X = 0) + 1^2 \cdot P(X = 1)) - p^2 \\ &= p - p^2 = p(1 - p) \end{aligned}$$

Geometric Random Variable

A Geometric random variable describes the number of (identical) Bernoulli trials that occur before the first success is observed. The distribution has a single parameter, the probability of a success p. There is another slightly different characterization that counts the number of failures before the first success. We will focus on the former for now.

X	$P(X=x)$
1	p
2	$p(1-p)$
3	$p(1-p)^2$
4	$p(1-p)^3$
$\vdots$	$\vdots$

$$P(X=x|p) = p(1-p)^{x-1}$$

Some useful infinite sum results

For $|r|<1$ then,

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r} \qquad \sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2}$$

We can use the first result to show that X has a valid probability distribution,

$$\begin{aligned}
\sum_{x=1}^{\infty} P(X=x) &= \sum_{x=1}^{\infty} p(1-p)^{x-1} = p\sum_{x=1}(1-p)^{x-1} \\
&= \frac{p}{(1-p)}\sum_{x=1}(1-p)^x = \frac{p}{(1-p)}\left(\frac{1}{1-(1-p)} - 1\right) \\
&= \left(\frac{1}{1-p} - \frac{p}{(1-p)}\right) \\
&= \frac{1-p}{1-p} = 1
\end{aligned}$$

Properties of a Geometric Random Variable

Let $X \sim \text{Geo}(p)$ then

$$\begin{aligned}
E(X) &= \sum_{x=1}^{\infty} x\; P(X=x) = \sum_{x=1}^{\infty} x\; p(1-p)^{x-1} \\
&= \frac{p}{(1-p)}\sum_{x=1}^{\infty} x\,(1-p)^x = \frac{p}{(1-p)}\frac{(1-p)}{(1-(1-p))^2} \\
&= 1/p
\end{aligned}$$

$$Var(X) = \frac{1-p}{p^2}$$

Combinations

A common problem in probability asks - if we have n items and want to select k of them how many possible groupings (order does not matter) are there?

Given by the binomial coefficient

$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

How many combinations of two numbers between 1 and 6 are there:

Permutations

Another option for those n items is if we select k of them and want to know how many possible unique orderings there are.

Given by

$$\frac{n!}{(n-k)!}$$

How many permutations of two numbers between 1 and 6 are there:

Discrete RVs Binomial RVs

Derivation

Discrete RVs Binomial RVs

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 5 15 20 15 5 1

1 6 20 35 35 20 6 1

1 7 26 55 70 55 26 7 1

⋮

$\binom{0}{0}$

$\binom{1}{0}$ $\binom{1}{1}$

$\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

$\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

$\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

Discrete RVs Binomial RVs

Some properties of the Combinations / Binomial coefficient

$$\binom{n}{k} = \binom{n}{n-k}$$

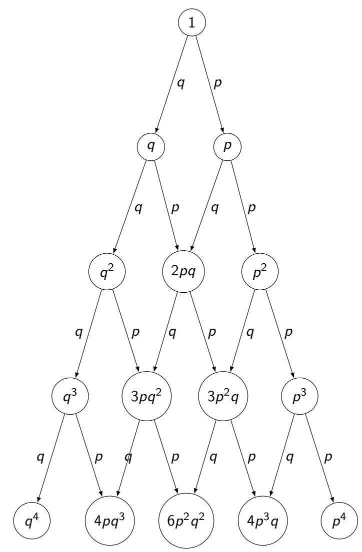
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}, \text{ for } 0 < k < n$$

Example - Cell Culture

A researcher is working with a new cell line, if there is a 10% chance of a single culture becoming contaminated during the week what is the probability that if the researcher has four cultures that only one of them will be contaminated at the end of the week? What about the probability k cultures lasting the week?

Binomial Distribution



Binomial Distribution

We define a random variable X that reflects the *number of successes* in a *fixed number* of *independent trials* with the *same probability of success* as having a binomial distribution.

If there are n trials then

$$X \sim \text{Binom}(n, p)$$

$$P(X = k|n, p) = f(k|n, p) = \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial theorem

Another useful result (and a connection with combinations) is the Binomial theorem which states:

$$(a+b)^m = \sum_{k=0}^{m} \binom{n}{m} x^m \; y^{n-m}$$

Properties of Binomial RVs

Let $X \sim \text{Binom}(n, p)$ then,

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{n} x \; P(X=x) = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} x \frac{n!}{(n-x)!x!} p^x (1-p)^{n-x} = \sum_{x=1}^{n} \frac{n!}{(n-x)!(x-1)!} p^x (1-p)^{n-x} \\
&= np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!(x-1)!} p^{x-1} (1-p)^{n-x} \\
&= np \sum_{x'=0}^{n-1} \frac{(n-1)!}{(n-(x'+1))!(x')!} p^{x'} (1-p)^{n-(x'+1)} \\
&= np \sum_{x'=0}^{n-1} \frac{(n-1)!}{(n-1-x')!(x')!} p^{x'} (1-p)^{n-1-x'} \\
&= np(p+(1-p))^n = np
\end{aligned}
$$

Properties of Binomial RVs

Let $X \sim \text{Binom}(n, p)$ then,

$$
\begin{aligned}
Var(X) &= E\left[(X - E(X))^2\right] \\
&= \sum_{x=0}^{n} (x - np)^2 \; P(X=x) \\
&= \sum_{x=0}^{n} (x - np)^2 \; P(X=x) \binom{n}{x} p^x (1-p)^{n-x} \\
&\;\vdots \qquad \text{(lots of awfulness)} \\
&= np(1-p)
\end{aligned}
$$

We'll see an simple and elegant way of solving this on Wednesday.

St. Petersburg Lottery

We start with \$1 on the table and a coin.

At each step: Toss the coin; if it shows Heads, take the money. If it shows Tails, I double the money on the table.

How much would you pay me to play this game? i.e. what is the expected value?