Lecture 20 - Introduction to Multiple Regression

Sta102 / BME102

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November 16, 2015

Poverty vs. region (east, west)

```
str(poverty)

## 'data.frame': 51 obs. of  8 variables:
##  $ State                 : Factor w/ 51 levels "Alabama","Alaska",..: 1 2 3 4 5 6 7
##  $ Metropolitan.Residence: num  55.4 65.6 88.2 52.5 94.4 84.5 87.7 80.1 100 89.3 ...
##  $ Caucasian             : num  71.3 70.8 87.7 81 77.5 90.2 85.4 76.3 36.2 80.6 ...
##  $ Graduates             : num  79.9 90.6 83.8 80.9 81.1 88.7 87.5 88.7 86 84.7 ...
##  $ Poverty               : num  14.6 8.3 13.3 18 12.8 9.4 7.8 8.1 16.8 12.1 ...
##  $ PercFemaleHH          : num  14.2 10.8 11.1 12.1 12.6 9.6 12.1 13.1 18.9 12 ...
##  $ region2               : Factor w/ 2 levels "east","west": 1 2 2 2 2 2 1 1 1 ...
##  $ region4               : Factor w/ 4 levels "northeast","midwest",..: 4 3 3 4 3 3
```


Poverty vs. region (east, west)

```
by(poverty$Poverty, poverty$region2,
   function(x) c(mean=mean(x),med=median(x),sd=sd(x),iqr=IQR(x)))

## poverty$region2: east
##      mean       med        sd       iqr
## 11.170370 10.300000  3.085427  4.600000
## --------------------------------------------
## poverty$region2: west
##      mean       med        sd       iqr
## 11.550000 10.700000  3.168459  4.000000
```


Poverty vs. region (east, west)

```
summary(lm(Poverty ~ region2, data=poverty))

##
## Call:
## lm(formula = Poverty ~ region2, data = poverty)
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -5.5704 -2.2000 -0.8704  2.0398  6.4500
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  11.1704     0.6013  18.576   <2e-16 ***
## region2west   0.3796     0.8766   0.433    0.667
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 3.125 on 49 degrees of freedom
## Multiple R-squared:  0.003813,Adjusted R-squared:  -0.01652
## F-statistic: 0.1875 on 1 and 49 DF,  p-value: 0.6669
```

Poverty vs. region (east, west)

$$\widehat{\%\ poverty} = 11.17 + 0.38 \times \mathbb{1}_{west}$$

- *Explanatory variable*: region
- *Reference level*: east
- *Intercept:* estimated average % poverty in eastern states is 11.17%
  - This is the value we get if we plug in *0* for the explanatory variable
- *Slope:* estimated average % poverty in western states is 0.38% higher than eastern states.
  - Estimated average % poverty in western states is 11.17 + 0.38 = 11.55%.
  - This is the value we get if we plug in *1* for the explanatory variable

Poverty vs. Region (Northeast, Midwest, West, South)

```
summary(lm(Poverty ~ region4, data=poverty))

##
## Call:
## lm(formula = Poverty ~ region4, data = poverty)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -6.359 -1.559 -0.025  1.574  6.508
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)      9.5000     0.8682  10.943 1.62e-14 ***
## region4midwest   0.0250     1.1485   0.022 0.982725
## region4west      1.7923     1.1294   1.587 0.119220
## region4south     4.1588     1.0736   3.874 0.000331 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.604 on 47 degrees of freedom
## Multiple R-squared:  0.3361,Adjusted R-squared:  0.2938
## F-statistic: 7.933 on 3 and 47 DF,  p-value: 0.0002205
```


Poverty vs. Region (Northeast, Midwest, West, South)

Which region (Northeast, Midwest, West, South) is the reference level?

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	9.50	0.87	10.94	0.00
region4midwest	0.03	1.15	0.02	0.98
region4west	1.79	1.13	1.59	0.12
region4south	4.16	1.07	3.87	0.00

Interpretation:

- Predict 9.50% poverty in Northeast
- Predict 9.53% poverty in Midwest
- Predict 11.29% poverty in West
- Predict 13.66% poverty in South

Poverty vs. Region (Northeast, Midwest, West, South)

```
by(poverty$Poverty, poverty$region4,
   function(x) c(mean=mean(x),med=median(x),sd=sd(x),iqr=IQR(x)))

## poverty$region4: northeast
##     mean      med       sd      iqr
## 9.500000 9.600000 2.381701 2.500000
## --------------------------------------------
## poverty$region4: midwest
##     mean      med       sd      iqr
## 9.525000 9.550000 1.415579 1.550000
## --------------------------------------------
## poverty$region4: west
##      mean       med        sd       iqr
## 11.292308 10.800000  2.647471  3.400000
## --------------------------------------------
## poverty$region4: south
##      mean       med        sd       iqr
## 13.658824 14.200000  3.233431  3.900000
```

Poverty vs. Region (Northeast, Midwest, West, South)

```
summary(aov(poverty$Poverty~poverty$region4))

##                  Df Sum Sq Mean Sq F value  Pr(>F)
## poverty$region4  3  161.4   53.81   7.933 0.00022 ***
## Residuals       47  318.8    6.78
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```


Poverty vs. Region (Northeast, Midwest, West, South)

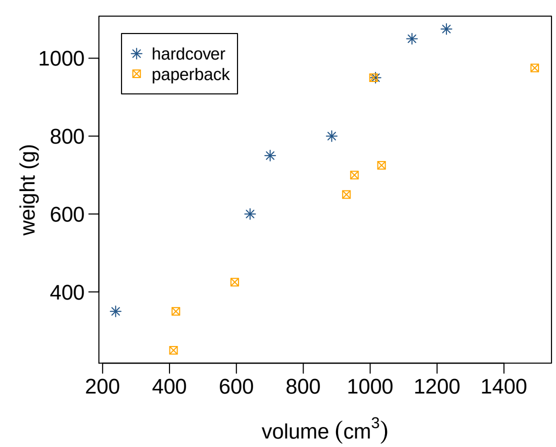
```
summary(lm(Poverty ~ region4, data=poverty))

##
## Call:
## lm(formula = Poverty ~ region4, data = poverty)
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -6.359 -1.559 -0.025  1.574  6.508
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)      9.5000     0.8682  10.943 1.62e-14 ***
## region4midwest   0.0250     1.1485   0.022 0.982725
## region4west      1.7923     1.1294   1.587 0.119220
## region4south     4.1588     1.0736   3.874 0.000331 ***
## ---
## Signif. codes:
## 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 2.604 on 47 degrees of freedom
## Multiple R-squared:  0.3361,Adjusted R-squared:  0.2938
## F-statistic: 7.933 on 3 and 47 DF,  p-value: 0.0002205
```


Weights of books

	weight (g)	volume (cm^3)	cover
1	800	885	hc
2	950	1016	hc
3	1050	1125	hc
4	350	239	hc
5	750	701	hc
6	600	641	hc
7	1075	1228	hc
8	250	412	pb
9	700	953	pb
10	650	929	pb
11	975	1492	pb
12	350	419	pb
13	950	1010	pb
14	425	595	pb
15	725	1034	pb

Weights of hard cover and paperback books

Can you identify a trend in the relationship between volume and weight of hardcover and paperback books?



Modeling weights of books using volume and cover type

```
book_mlr = lm(weight ~ volume + cover, data = allbacks)
summary(book_mlr)

## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)  197.96284   59.19274   3.344 0.005841 **
## volume         0.71795    0.06153  11.669  6.6e-08 ***
## cover:pb    -184.04727   40.49420  -4.545 0.000672 ***
##
##
## Residual standard error: 78.2 on 12 degrees of freedom
## Multiple R-squared: 0.9275, Adjusted R-squared: 0.9154
## F-statistic: 76.73 on 2 and 12 DF,  p-value: 1.455e-07
```


Linear model

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	197.96	59.19	3.34	0.01
volume	0.72	0.06	11.67	0.00
cover:pb	-184.05	40.49	-4.55	0.00

$$\widehat{\texttt{weight}} = 197.96 + 0.72 \texttt{ volume} - 184.05 \texttt{ cover:pb}$$

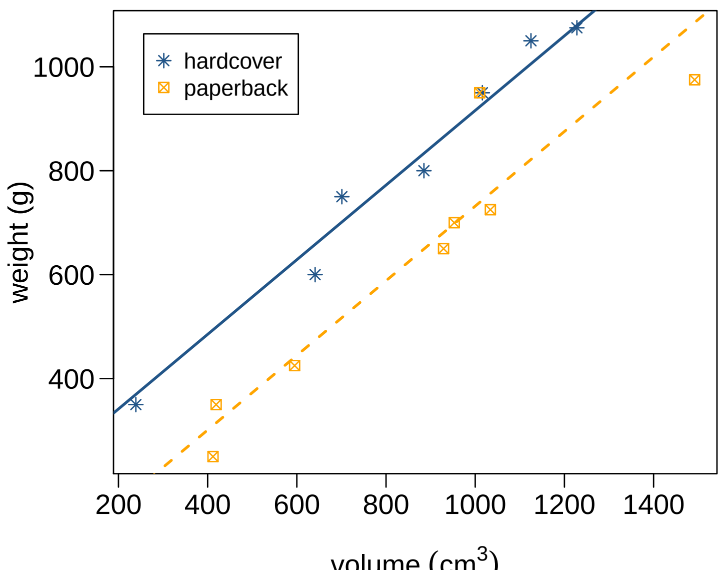
1. For *hardcover* books: plug in *0* for cover

$$\begin{aligned}\widehat{\texttt{weight}} &= 197.96 + 0.72 \texttt{ volume} - 184.05 \times 0 \\ &= 197.96 + 0.72 \texttt{ volume}\end{aligned}$$

2. For *paperback* books: plug in *1* for cover

$$\begin{aligned}\widehat{\texttt{weight}} &= 197.96 + 0.72 \texttt{ volume} - 184.05 \times 1 \\ &= 13.91 + 0.72 \texttt{ volume}\end{aligned}$$

Visualising the linear model



Interpretation of the regression coefficients

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	197.96	59.19	3.34	0.01
volume	0.72	0.06	11.67	0.00
cover:pb	-184.05	40.49	-4.55	0.00

- *Slope of volume:* All else held constant, for each 1 cm^3 increase in volume we would expect weight to increase on average by 0.72 grams.
- *Slope of cover:* All else held constant, the model predicts that paperback books weigh 184 grams less than hardcover books, on average.
- *Intercept:* Hardcover books with no volume are expected on average to weigh 198 grams.
  - Obviously, the intercept does not make sense in context. It only serves to adjust the height of the line.

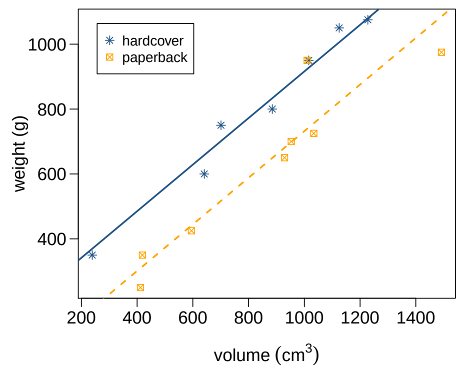
Prediction

What is the correct calculation for the predicted weight of a paperback book that has a volume of 600 cm^3?

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	197.96	59.19	3.34	0.01
volume	0.72	0.06	11.67	0.00
cover:pb	-184.05	40.49	-4.55	0.00

A note on interactions

$$\widehat{\texttt{weight}} = 197.96 + 0.72\ \texttt{volume} - 184.05\ \texttt{cover:pb}$$



This model assumes that hardcover and paperback books have the same slope for the relationship between their volume and weight. If this isn't reasonable, then we would include an "interaction" variable in the model.

Example of an interaction

```
summary( lm(weight ~ volume + cover + volume:cover, data = allbacks) )

## Coefficients:
##                  Estimate Std. Error t value Pr(>|t|)
## (Intercept)     161.58654   86.51918   1.868   0.0887 .
## volume            0.76159    0.09718   7.837 7.94e-06 ***
## coverpb        -120.21407  115.65899  -1.039   0.3209
## volume:coverpb   -0.07573    0.12802  -0.592   0.5661
##
## Residual standard error: 80.41 on 11 degrees of freedom
## Multiple R-squared: 0.9297, Adjusted R-squared: 0.9105
## F-statistic:  48.5 on 3 and 11 DF,  p-value: 1.245e-06
```

$$\widehat{\texttt{weight}} = 161.58 + 0.76\ \texttt{volume} - 120.21\ \texttt{cover:pb} - 0.076\ \texttt{volume} \times \texttt{cover:pb}$$

Example of an interaction - interpretation

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	161.5865	86.5192	1.87	0.0887
volume	0.7616	0.0972	7.84	0.0000
coverpb	-120.2141	115.6590	-1.04	0.3209
volume:coverpb	-0.0757	0.1280	-0.59	0.5661

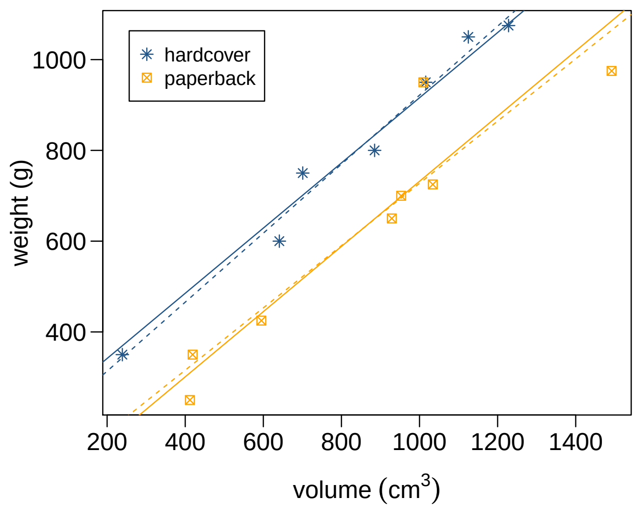
Regression equations for hardbacks:

$$\begin{aligned}\widehat{\texttt{weight}} &= 161.58 + 0.76\ \texttt{volume} - 120.21 \times \mathit{0} - 0.076\ \texttt{volume} \times \mathit{0} \\ &= 161.58 + 0.76\ \texttt{volume}\end{aligned}$$

Regression equations for paperbacks:

$$\begin{aligned}\widehat{\texttt{weight}} &= 161.58 + 0.76\ \texttt{volume} - 120.21 \times \mathit{1} - 0.076\ \texttt{volume} \times \mathit{1} \\ &= 41.37 + 0.686\ \texttt{volume}\end{aligned}$$

Example of an interaction - Results



Another look at R

For a linear regression we have defined the correlation coefficient to be

$$R = \text{Cor}(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

This definition works fine for the simple linear regression case where X and Y are numeric variables, but does not work for regression with a categorical predictor or for multiple regression.

A more useful, and equivalent, definition is $R = \text{Cor}(Y, \hat{Y})$, which will work for all regression examples we will see in this class.

Another look at R, cont.

Claim: $\text{Cor}(X, Y) = \text{Cor}(Y, \hat{Y})$

Remember: $\text{Cor}(X, Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y}$, $\quad \hat{Y} = b_0 + b_1\, X$,
$\text{Var}(aX + b) = a^2\, \text{Var}(X)$,
$\text{Cov}(aX + b, Y) = a\, \text{Cov}(X, Y)$

$$\begin{aligned}
\text{Cor}(Y, \hat{Y}) &= \frac{\text{Cov}(Y, \hat{Y})}{\sqrt{\text{Var}(Y)\text{Var}(\hat{Y})}} \\
&= \frac{\text{Cov}(Y, b_0 + b_1\, X)}{\sqrt{\sigma_Y^2 \text{Var}(b_0 + b_1\, X)}} \\
&= \frac{b_1\, \text{Cov}(Y, X)}{\sigma_Y \sqrt{b_1^2 \text{Var}(X)}} \\
&= \frac{b_1\, \text{Cov}(Y, X)}{b_1\, \sigma_Y \sigma_X} \\
&= \text{Cor}(X, Y)
\end{aligned}$$

Another look at R^2

So how can we claim that R^2 is a measure of variability "explained" by the model?

Remember, in an ANOVA we can partition total uncertainty into model (group) uncertainty and residual (error) uncertainty.

$$SST = SSG + SSE$$

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \bar{y})^2 = \sum_{i=1}^{k} \sum_{j=1}^{n_i} (\bar{y}_i - \bar{y})^2 + \sum_{i=1}^{k} \sum_{j=1}^{n_i} (y_{ij} - \bar{y}_i)^2$$

For a regression we can do the same thing, just replacing $\bar{y}_i$ with $\hat{y}_i$

$$SST = SSR + SSE$$

$$\sum_{i=1}^{n} (y_i - \bar{y})^2 = \sum_{i=1}^{n} (\hat{y}_i - \bar{y})^2 + \sum_{i=1}^{n} (y_i - \hat{y}_i)^2$$

Another look at R^2

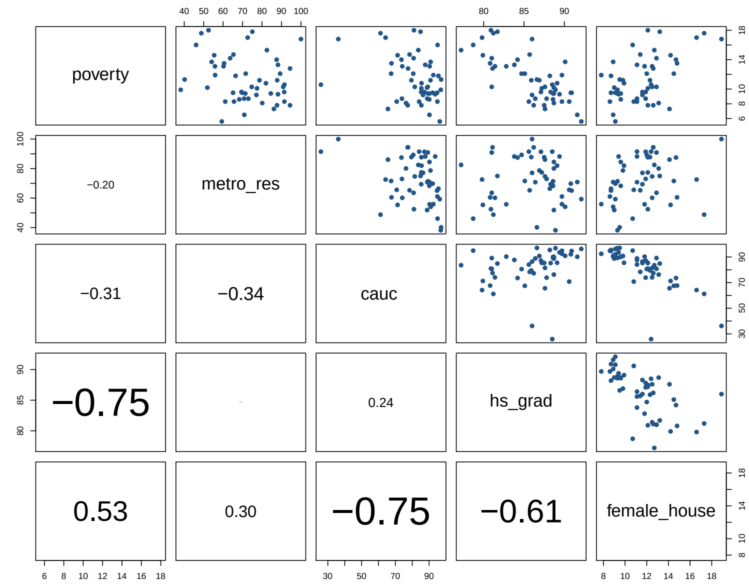
After a fair bit of algebra we can show that,

$$R^2 = \text{Cor}(Y, \hat{Y})^2 = \frac{Cov(Y, \hat{Y})^2}{Var(Y)Var(\hat{Y})}$$

$$= \frac{\sum_{i=1}^{n}(\hat{Y}_i - \bar{Y})^2}{\sum_{i=1}^{n}(Y_i - \bar{Y})^2} = \frac{SSR}{SST}$$

$$= \frac{SST - SSE}{SST} = 1 - \frac{SSE}{SST}$$

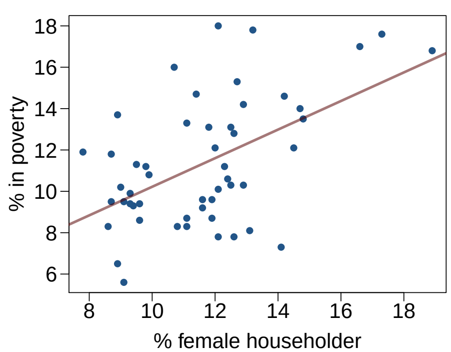
Revisit: Modeling poverty



Predicting poverty using % female householder

```
summary(lm(poverty ~ female_house, data = poverty))
```

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	3.31	1.90	1.74	0.09
female_house	0.69	0.16	4.32	0.00



$R = 0.53$

$R^2 = 0.53^2 = 0.28$

Another look at R^2 - from last week

```
anova(lm(poverty ~ female_house, data = poverty))
```

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
female_house	1	132.57	132.57	18.68	0.00
Residuals	49	347.68	7.10		
Total	50	480.25			

$$SS_{Tot} = \sum (y - \bar{y})^2 = 480.25 \rightarrow \textit{total variability}$$

$$SS_{Err} = \sum e_i^2 = 347.68 \rightarrow \textit{unexplained variability}$$

$$SS_{Reg} = SS_{Total} - SS_{Error} \rightarrow \textit{explained variability}$$

$$= 480.25 - 347.68 = 132.57$$

$$R^2 = \frac{\text{explained variability}}{\text{total variability}} = \frac{132.57}{480.25} = 0.28 \checkmark$$

Predicting poverty using % female hh + % cauc

```
pov_mlr = lm(poverty ~ female_house + cauc, data = poverty)
summary(pov_mlr)
```

	Estimate	Std. Error	t value	Pr(>|t|)
(Intercept)	-2.58	5.78	-0.45	0.66
female_house	0.89	0.24	3.67	0.00
cauc	0.04	0.04	1.08	0.29

```
anova(pov_mlr)
```

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
female_house	1	132.57	132.57	18.74	0.00
cauc	1	8.21	8.21	1.16	0.29
Residuals	48	339.47	7.07		
Total	50	480.25			

$$R^2 = \frac{\text{explained variability}}{\text{total variability}} = \frac{132.57 + 8.21}{480.25} = 0.29$$

R^2 vs. adjusted R^2

	R^2	Adjusted R^2
Model 1 (poverty vs. female_house)	0.2760	0.2613
Model 2 (poverty vs. female_house + cauc)	0.2931	0.2637

- We would like to have some criteria to evaluate if adding an additional variable makes a difference in the explanatory power of the model.
- When any variable is added to the model R^2 increases.
- Adjusted R^2 is based on R^2 but it penalizes the addition of variables.

Adjusted R^2

Adjusted R^2

$$R^2_{adj} = 1 - \left(\frac{SS_{Error}}{SS_{Total}} \times \frac{n-1}{n-k-1} \right)$$

where n is the number of cases and k is the number of predictors (explanatory variables excluding the intercept) in the model.

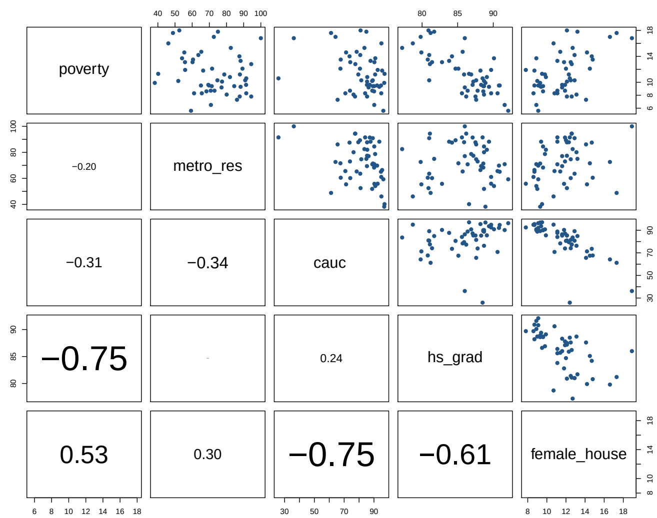
- Because k is never negative, R^2_{adj} will always be less than or equal to R^2.
- R^2_{adj} applies a penalty for the number of predictors included in the model.
- Therefore, we prefer models with higher R^2_{adj}

Calculate adjusted R^2

	Df	Sum Sq	Mean Sq	F value	Pr(>F)
female_house	1	132.57	132.57	18.74	0.0001
cauc	1	8.21	8.21	1.16	0.2868
Residuals	48	339.47	7.07		
Total	50	480.25			

$$\begin{aligned} R^2_{adj} &= 1 - \left(\frac{SS_{Error}}{SS_{Total}} \times \frac{n-1}{n-k-1} \right) \\ &= 1 - \left(\frac{339.47}{480.25} \times \frac{51-1}{51-2-1} \right) \\ &= 1 - \left(\frac{339.47}{480.25} \times \frac{50}{48} \right) \\ &= 1 - 0.74 \\ &= 0.26 \end{aligned}$$

We saw that adding the variable cauc to the model only marginally increased adjusted R^2, i.e. did not add much useful information to the model. Why?



Collinearity between explanatory variables (cont.)

- Two predictor variables are said to be collinear when they are correlated, and this *collinearity* (also called *multicollinearity*) complicates model estimation.

  Remember: *Predictors are also called explanatory or independent variables, so ideally they should be independent of each other.*

- We don't like adding predictors that are associated with each other to the model, because often times the addition of such variable brings nothing to the table. Instead, we prefer the simplest model that explains as much as possible - the most *parsimonious* model.
- In addition, inclusion of collinear variables can result in biased estimates of the slope parameters.
- While it's impossible to avoid all collinearity, often experiments are designed to control for correlated predictors.