

## Lecture 22 - Logistic Regression

Sta102 / BME102

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Background

## Regression so far ...

At this point we have covered:

- Simple linear regression
  - Relationship between numerical response and a numerical or categorical predictor
- Multiple regression
  - Relationship between numerical response and multiple numerical and/or categorical predictors

What we haven't seen is what to do when the predictors are weird (nonlinear, complicated dependence structure, etc.) or when the response is weird (categorical, count data, etc.)

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Background

## Recap of what you should know how to do ...

- Model parameter interpretation
- Hypothesis tests for regression parameters
- Confidence intervals for regression parameters
- Hypothesis tests for the complete model
- Confidence and prediction intervals for predicted means and values (SLR only)
- Model diagnostics, residuals plots, outliers
- $R^2$ , Adjusted  $R^2$
- Model selection (MLR only)
- Simple transformations

Background

## Odds

Odds are another way of quantifying the probability of an event, commonly used in gambling (and logistic regression).

### Odds

For some event  $E$ ,

$$\text{odds}(E) = \frac{P(E)}{P(E^c)} = \frac{P(E)}{1 - P(E)}$$

Similarly, if we are told the odds of  $E$  are  $x$  to  $y$  then

$$\text{odds}(E) = \frac{x}{y} = \frac{x/(x+y)}{y/(x+y)}$$

which implies

$$P(E) = x/(x+y), \quad P(E^c) = y/(x+y)$$

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## Example - Donner Party

In 1846 the Donner and Reed families left Springfield, Illinois, for California by covered wagon. In July, the Donner Party, as it became known, reached Fort Bridger, Wyoming. There its leaders decided to attempt a new and untested route to the Sacramento Valley. Having reached its full size of 87 people and 20 wagons, the party was delayed by a difficult crossing of the Wasatch Range and again in the crossing of the desert west of the Great Salt Lake. The group became stranded in the eastern Sierra Nevada mountains when the region was hit by heavy snows in late October. By the time the last survivor was rescued on April 21, 1847, 40 of the 87 members had died from famine and exposure to extreme cold.

From Ramsey, Schafer (2002). *The Statistical Sleuth*

## Example - Donner Party - Data

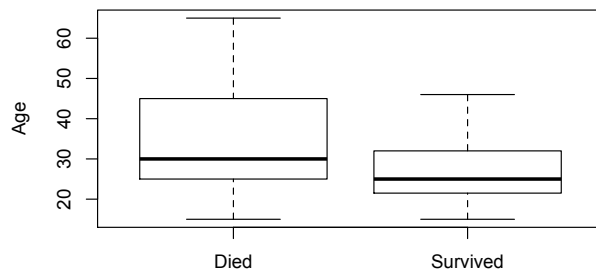
	Age	Sex	Status
1	23.00	Male	Died
2	40.00	Female	Survived
3	40.00	Male	Survived
4	30.00	Male	Died
5	28.00	Male	Died
⋮	⋮	⋮	⋮
43	23.00	Male	Survived
44	24.00	Male	Died
45	25.00	Female	Survived

## Example - Donner Party - EDA

Status vs. Gender:

	Male	Female
Died	20	5
Survived	10	10

Status vs. Age:



## Example - Donner Party - ???

It seems clear that both age and gender have an effect on someone's survival, how do we come up with a model that will let us explore this relationship?

Even if we set Died to 0 and Survived to 1, this isn't something we can transform our way out of - we need something more.

One way to think about the problem - we can treat Survived and Died as successes and failures arising from a Bernoulli trial where the probability of a success is given by a transformation of a linear model of the predictors.

## Generalized linear models

It turns out that this is a very general way of addressing this type of problem in regression, and the resulting models are called generalized linear models (GLMs). Logistic regression is just one example of this type of model.

All generalized linear models have the following three characteristics:

- 1 A probability distribution describing the outcome variable
- 2 A linear model

$$\eta = \beta_0 + \beta_1 X_1 + \dots + \beta_n X_n$$

- 3 A link function that relates the linear model to the parameter of the outcome distribution

$$g(p) = \eta \text{ or } p = g^{-1}(\eta)$$

## Logistic Regression

Logistic regression is a GLM used to model a binary categorical variable using numerical and categorical predictors.

We assume a binomial distribution produced the outcome variable and we therefore want to model  $p$  the probability of success for a given set of predictors.

To finish specifying the Logistic model we just need to establish a reasonable link function that connects  $\eta$  to  $p$ . There are a variety of options but the most commonly used is the logit function.

Logit function

$$\text{logit}(p) = \log\left(\frac{p}{1-p}\right), \text{ for } 0 \leq p \leq 1$$

## Properties of the Logit

The logit function takes a value between 0 and 1 and maps it to a value between  $-\infty$  and  $\infty$ .

Inverse logit (logistic) function

$$g^{-1}(x) = \frac{\exp(x)}{1 + \exp(x)} = \frac{1}{1 + \exp(-x)}$$

The inverse logit function takes a value between  $-\infty$  and  $\infty$  and maps it to a value between 0 and 1.

This formulation is also useful for interpreting the model, since the logit can be interpreted as the log odds of a success - more on this later.

## The logistic regression model

The three GLM criteria give us:

$$y_i \sim \text{Bern}(p_i)$$

$$\eta_i = \beta_0 + \beta_1 x_{1,i} + \dots + \beta_n x_{n,i}$$

$$\text{logit}(p_i) = \eta_i$$

From which we get,

$$p_i = \frac{\exp(\beta_0 + \beta_1 x_{1,i} + \dots + \beta_n x_{n,i})}{1 + \exp(\beta_0 + \beta_1 x_{1,i} + \dots + \beta_n x_{n,i})}$$

## Example - Donner Party - Model

In R we fit a GLM in the same way as a linear model except we use `glm` instead of `lm`. (We specify the type of GLM to fit using the family argument)

```
summary(glm(Status ~ Age, data=donner, family=binomial))
```

```
## Call:
## glm(formula = Status ~ Age, family = binomial, data = donner)
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  1.81852    0.99937   1.820  0.0688 .
## Age         -0.06647    0.03222  -2.063  0.0391 *
##
## Null deviance: 61.827  on 44  degrees of freedom
## Residual deviance: 56.291  on 43  degrees of freedom
## AIC: 60.291
##
## Number of Fisher Scoring iterations: 4
```

## Example - Donner Party - Prediction

	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	1.8185	0.9994	1.82	0.0688
Age	-0.0665	0.0322	-2.06	0.0391

Model:

$$\log\left(\frac{p}{1-p}\right) = 1.8185 - 0.0665 \times \text{Age}$$

Odds / Probability of survival for a newborn (Age=0):

$$\begin{aligned} \log\left(\frac{p}{1-p}\right) &= 1.8185 - 0.0665 \times 0 \\ \frac{p}{1-p} &= \exp(1.8185) = 6.16 \\ p &= 6.16 / 7.16 = 0.86 \end{aligned}$$

## Example - Donner Party - Prediction (cont.)

Model:

$$\log\left(\frac{p}{1-p}\right) = 1.8185 - 0.0665 \times \text{Age}$$

Odds / Probability of survival for a 25 year old:

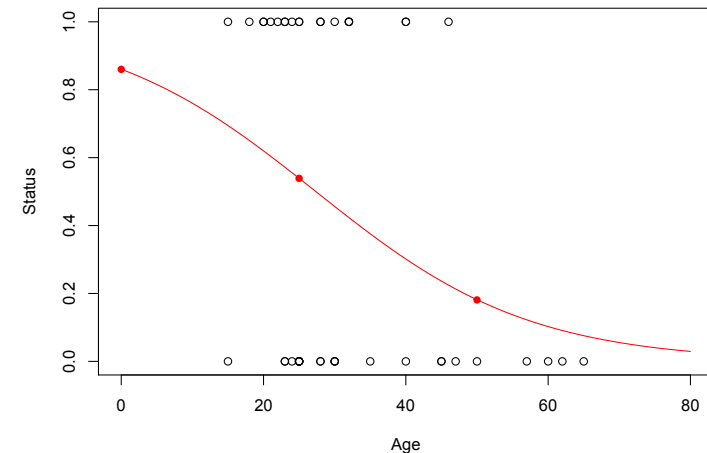
$$\begin{aligned} \log\left(\frac{p}{1-p}\right) &= 1.8185 - 0.0665 \times 25 \\ \frac{p}{1-p} &= \exp(0.156) = 1.17 \\ p &= 1.17 / 2.17 = 0.539 \end{aligned}$$

Odds / Probability of survival for a 50 year old:

$$\begin{aligned} \log\left(\frac{p}{1-p}\right) &= 1.8185 - 0.0665 \times 50 \\ \frac{p}{1-p} &= \exp(-1.5065) = 0.222 \\ p &= 0.222 / 1.222 = 0.181 \end{aligned}$$

## Example - Donner Party - Prediction (cont.)

$$\log\left(\frac{p}{1-p}\right) = 1.8185 - 0.0665 \times \text{Age}$$



## Example - Donner Party - Interpretation

	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	1.8185	0.9994	1.82	0.0688
Age	-0.0665	0.0322	-2.06	0.0391

Simple interpretation is only possible in terms of *log odds* and *log odds ratios* for intercept and slope terms.

*Intercept*: The *log odds* of survival for a party member with an age of 0. From this we can calculate the odds or probability, but additional calculations are necessary.

*Slope*: For a unit increase in age (being 1 year older) how much will the *log odds ratio* change, not particularly intuitive. More often than not we care only about sign and relative magnitude.

## Example - Donner Party - Interpretation - Slope

$$\begin{aligned}\log\left(\frac{p_1}{1-p_1}\right) &= 1.8185 - 0.0665(x+1) \\ &= 1.8185 - 0.0665x - 0.0665\end{aligned}$$

$$\log\left(\frac{p_2}{1-p_2}\right) = 1.8185 - 0.0665x$$

$$\log\left(\frac{p_1}{1-p_1}\right) - \log\left(\frac{p_2}{1-p_2}\right) = -0.0665$$

$$\log\left(\frac{p_1}{1-p_1} / \frac{p_2}{1-p_2}\right) = -0.0665$$

$$\frac{p_1}{1-p_1} / \frac{p_2}{1-p_2} = \exp(-0.0665) = 0.94$$

## Example - Donner Party - Age and Gender

```
summary(glm(Status ~ Age + Sex, data=donner, family=binomial))

## Call:
## glm(formula = Status ~ Age + Sex, family = binomial, data = donner)
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  1.63312    1.11018   1.471   0.1413
## Age          -0.07820    0.03728  -2.097   0.0359 *
## SexFemale    1.59729    0.75547   2.114   0.0345 *
## ---
##
## (Dispersion parameter for binomial family taken to be 1)
##
##    Null deviance: 61.827  on 44  degrees of freedom
## Residual deviance: 51.256  on 42  degrees of freedom
## AIC: 57.256
##
## Number of Fisher Scoring iterations: 4
```

*Gender slope*: When the other predictors are held constant this is the log odds ratio between the contrast (Female) and the reference level (Male).

## Example - Donner Party - Gender Models

Just like MLR we can plug in gender to arrive at two status vs age models for men and women respectively.

General model:

$$\log\left(\frac{p_1}{1-p_1}\right) = 1.63312 + -0.07820 \times \text{Age} + 1.59729 \times \text{Sex}$$

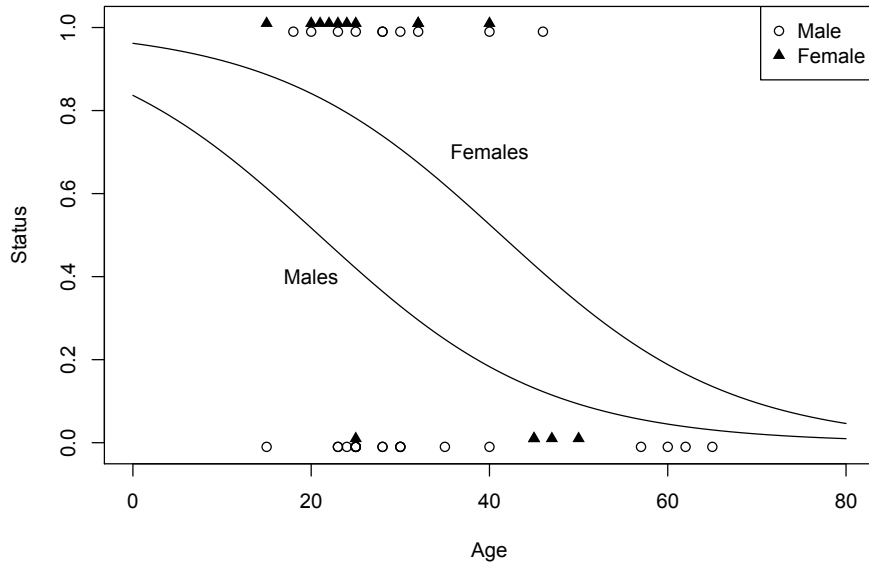
Male model:

$$\begin{aligned}\log\left(\frac{p_1}{1-p_1}\right) &= 1.63312 + -0.07820 \times \text{Age} + 1.59729 \times 0 \\ &= 1.63312 + -0.07820 \times \text{Age}\end{aligned}$$

Female model:

$$\begin{aligned}\log\left(\frac{p_1}{1-p_1}\right) &= 1.63312 + -0.07820 \times \text{Age} + 1.59729 \times 1 \\ &= 3.23041 + -0.07820 \times \text{Age}\end{aligned}$$

## Example - Donner Party - Gender Models (cont.)



## Hypothesis test for the model

```
summary(glm(Status ~ Age + Sex, data=donner, family=binomial))

## Call:
## glm(formula = Status ~ Age + Sex, family = binomial, data = donner)
##
## Coefficients:
##             Estimate Std. Error z value Pr(>|z|)
## (Intercept)  1.63312   1.11018   1.471  0.1413
## Age         -0.07820   0.03728  -2.097  0.0359 *
## SexFemale    1.59729   0.75547   2.114  0.0345 *
## ---
##
## (Dispersion parameter for binomial family taken to be 1)
##
## Null deviance: 61.827  on 44  degrees of freedom
## Residual deviance: 51.256  on 42  degrees of freedom
## AIC: 57.256
##
## Number of Fisher Scoring iterations: 4
```

Note that the model output does not include any F-statistic, as a general rule there are not single model hypothesis tests for GLM models.

## Hypothesis tests for a coefficient

	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	1.6331	1.1102	1.47	0.1413
Age	-0.0782	0.0373	-2.10	0.0359
SexFemale	1.5973	0.7555	2.11	0.0345

We can still perform inference for individual coefficients, the basic framework is the same as SLR/MLR except we use a Z test instead of a t test.

Note the only tricky bit, which is beyond the scope of this course, is how the standard error is calculated.

## Testing for the slope of Age

	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	1.6331	1.1102	1.47	0.1413
Age	-0.0782	0.0373	-2.10	0.0359
SexFemale	1.5973	0.7555	2.11	0.0345

$$H_0 : \beta_{age} = 0$$

$$H_A : \beta_{age} \neq 0$$

$$Z = \frac{\hat{\beta}_{age} - \beta_{age}}{SE_{age}} = \frac{-0.0782 - 0}{0.0373} = -2.10$$

$$\begin{aligned} \text{p-value} &= P(|Z| > 2.10) = P(Z > 2.10) + P(Z < -2.10) \\ &= 2 \times 0.0178 = 0.0359 \end{aligned}$$

## Confidence interval for age slope coefficient

	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	1.6331	1.1102	1.47	0.1413
Age	-0.0782	0.0373	-2.10	0.0359
SexFemale	1.5973	0.7555	2.11	0.0345

Remember, the interpretation for a slope is the change in log odds ratio per unit change in the predictor.

Log odds ratio:

$$CI = PE \pm CV \times SE = -0.0782 \pm 1.96 \times 0.0373 = (-0.1513, -0.0051)$$

Odds ratio:

$$\exp(CI) = (\exp(-0.1513), \exp(-0.0051)) = (0.8596, 0.9949)$$

## Example - Birdkeeping and Lung Cancer

A 1972 - 1981 health survey in The Hague, Netherlands, discovered an association between keeping pet birds and increased risk of lung cancer. To investigate birdkeeping as a risk factor, researchers conducted a case-control study of patients in 1985 at four hospitals in The Hague (population 450,000). They identified 49 cases of lung cancer among the patients who were registered with a general practice, who were age 65 or younger and who had resided in the city since 1965. They also selected 98 controls from a population of residents having the same general age structure.

From Ramsey, F.L. and Schafer, D.W. (2002). *The Statistical Sleuth: A Course in Methods of Data Analysis* (2nd ed)

## Example - Birdkeeping and Lung Cancer - Data

	LC	FM	SS	BK	AG	YR	CD
1	LungCancer	Male	Low	Bird	37.00	19.00	12.00
2	LungCancer	Male	Low	Bird	41.00	22.00	15.00
3	LungCancer	Male	High	NoBird	43.00	19.00	15.00
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
147	NoCancer	Female	Low	NoBird	65.00	7.00	2.00

LC Whether subject has lung cancer

FM Sex of subject

SS Socioeconomic status

BK Indicator for birdkeeping

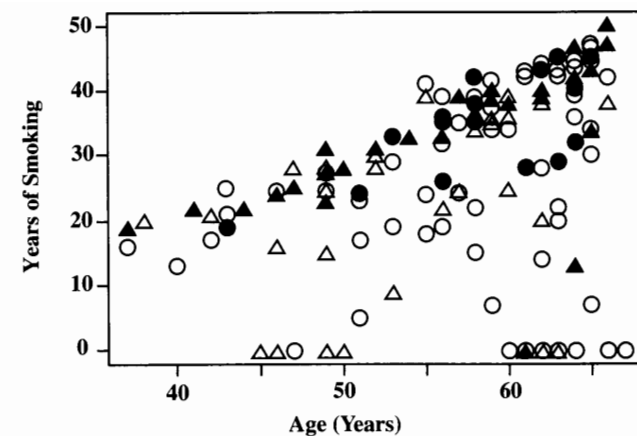
AG Age of subject (years)

YR Years of smoking prior to diagnosis or examination

CD Average rate of smoking (cigarettes per day)

**Note** - NoCancer is the reference response (0 or failure), LungCancer is the non-reference response (1 or success) - this matters for interpretation.

## Example - Birdkeeping and Lung Cancer - EDA



	Bird	No Bird
Lung Cancer	▲	●
No Lung Cancer	△	○

## Example - Birdkeeping and Lung Cancer - Model

```
summary(glm(LC ~ FM + SS + BK + AG + YR + CD, data=bird, family=binomial))

## Call:
## glm(formula = LC ~ FM + SS + BK + AG + YR + CD, family = binomial,
##      data = bird)
##
## Coefficients:
##              Estimate Std. Error z value Pr(>|z|)
## (Intercept) -1.93736    1.80425  -1.074  0.282924
## FMFemale    0.56127    0.53116   1.057  0.290653
## SSHigh     0.10545    0.46885   0.225  0.822050
## BKBird     1.36259    0.41128   3.313  0.000923 ***
## AG        -0.03976    0.03548  -1.120  0.262503
## YR         0.07287    0.02649   2.751  0.005940 **
## CD         0.02602    0.02552   1.019  0.308055
##
## (Dispersion parameter for binomial family taken to be 1)
##
##      Null deviance: 187.14  on 146  degrees of freedom
## Residual deviance: 154.20  on 140  degrees of freedom
## AIC: 168.2
##
## Number of Fisher Scoring iterations: 5
```

## Example - Birdkeeping and Lung Cancer - Interpretation

	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	-1.9374	1.8043	-1.07	0.2829
FMFemale	0.5613	0.5312	1.06	0.2907
SSHHigh	0.1054	0.4688	0.22	0.8221
BKBird	1.3626	0.4113	3.31	0.0009
AG	-0.0398	0.0355	-1.12	0.2625
YR	0.0729	0.0265	2.75	0.0059
CD	0.0260	0.0255	1.02	0.3081

Keeping all other predictors constant then,

- The odds ratio of getting lung cancer for bird keepers vs non-bird keepers is  $\exp(1.3626) = 3.91$ .
- The odds ratio of getting lung cancer for an additional year of smoking is  $\exp(0.0729) = 1.08$ .

*What do these numbers mean in practice?*

## What do the numbers not mean ...

The most common mistake made when interpreting logistic regression is to treat an odds ratio as a ratio of probabilities.

Bird keepers are *not* 4x more likely to develop lung cancer than non-bird keepers.

This is the difference between *relative risk* and an *odds ratio*.

$$RR = \frac{P(\text{disease}|\text{exposed})}{P(\text{disease}|\text{unexposed})}$$

$$OR = \frac{P(\text{disease}|\text{exposed})/[1 - P(\text{disease}|\text{exposed})]}{P(\text{disease}|\text{unexposed})/[1 - P(\text{disease}|\text{unexposed})]}$$

## Back to the birds

What is probability of lung cancer in a bird keeper if we knew that  $P(\text{lung cancer}|\text{no birds}) = 0.05$ ?

$$OR = \frac{P(\text{lung cancer}|\text{birds})/[1 - P(\text{lung cancer}|\text{birds})]}{P(\text{lung cancer}|\text{no birds})/[1 - P(\text{lung cancer}|\text{no birds})]}$$

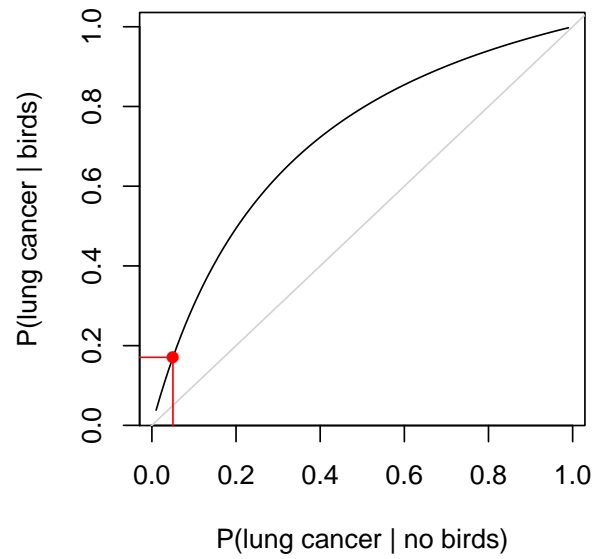
$$= \frac{P(\text{lung cancer}|\text{birds})/[1 - P(\text{lung cancer}|\text{birds})]}{0.05/[1 - 0.05]} = 3.91$$

$$P(\text{lung cancer}|\text{birds}) = \frac{3.91 \times \frac{0.05}{0.95}}{1 + 3.91 \times \frac{0.05}{0.95}} = 0.171$$

$$RR = P(\text{lung cancer}|\text{birds})/P(\text{lung cancer}|\text{no birds}) = 0.171/0.05 = 3.41$$



## Bird OR Curve



## OR Curves

