Lecture 4 - More Conditional Probability

Sta102 / BME102

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Basic Probability Review

$$0 \le P(A) \le 1$$

$$P(A^c) = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A|B) \times P(B) = P(B|A) \times P(A)$$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

Disjoint and Independent

$$\text{Independent} \not\Longrightarrow \text{Disjoint}$$
$$\text{Disjoint} \Longrightarrow \text{Dependent}$$
$$(\text{but Not Disjoint} \not\Longrightarrow \text{Dependent or Independent})$$

Your intuition can easily be wrong on this type of problem, always check using the definitions:

A and B are independent iff:

$$P(A \cap B) = P(A) \times P(B)$$

A and B are disjoint (or mutually exclusive) iff:

$$P(E \cap F) = 0$$

Other Important Terms

Joint probability of A and B	$P(A \cap B)$
Marginal probability of A	$P(A)$
Conditional probability of A given B	$P(A\|B)$

Sampling

Imagine an urn filled with white and black marbles
. . . or a deck of cards
. . . or a bingo cage
. . . or a hat full of raffle tickets

Two common options + one extra for completeness:

- Sampling without replacement
- Sampling with replacement
- Pólya's urn

Quick Examples

- What is the probability of being dealt two aces? (*Sampling without replacement*)

$$P(\text{draw an ace}) \times P(\text{draw an ace} \mid \text{already drawn 1 ace}) = \frac{4}{52}\frac{3}{51} = 0.0044$$

- What if you replace the first card before reshuffling and drawing the second card? (*Sampling with replacement*)

$$P(\text{draw an ace}) \times P(\text{draw an ace} \mid \text{already drawn 1 ace}) = \frac{4}{52}\frac{4}{52} = 0.0059$$

- What if you replace the first card and a copy before reshuffling and drawing the second card? (*Polya's Urn*)

$$P(\text{draw an ace}) \times P(\text{draw an ace} \mid \text{already drawn 1 ace}) = \frac{4}{52}\frac{5}{53} = 0.0073$$

- What is the probability of being dealt a royal flush in poker?

And now a brief magic trick ...

- What is the probability of being dealt a royal flush in poker? (Ace, king, queen, jack, 10 of the same suit)

$$P(A, K, Q, J, 10 \text{ of same suit}) = ?$$

This is a case where thinking about things as conditional probabilities (tree) actually makes things harder - easier to think about each step in terms of how many ways are left to get what we want.

$$\frac{20}{52} \times \frac{4}{51} \times \frac{3}{50} \times \frac{2}{49} \times \frac{1}{48} = 0.000000384$$

And now a brief magic trick ...

If you have ever shuffled a deck of cards you have done something no one else has ever done before or will ever do again ...

There are approximately 8×10^{67} possible configurations of a deck of 52 to cards

To put that in context:

- Cells in the human body (10^{14})
- Seconds since the big bag (10^{18})
- Grains of sand on all beaches on earch (7.5×10^{18})
- Stars in the universe (10^{23})
- Atoms in the observable universe (10^{80})
- A Googol (10^{100})

Birthday Problem

Ignoring leap years, and assuming birthdays are equally likely to be any day of the year, what is the probability that there is at least one shared birthday among the students in this class?

As of this morning there are 55 people enrolled in this course,

$$P(\text{at least one match}) = 1 - P(\text{no match}) = 1 - 0.0137 = 0.9863$$

Let A_i be the event that student i does not match any of the preceding students then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2|A_1)\cdots P(A_n|A_1,\ldots,A_{n-1})$$

Birthday Problem, cont.

Calculation:

$$P(A_1) = 365/365$$

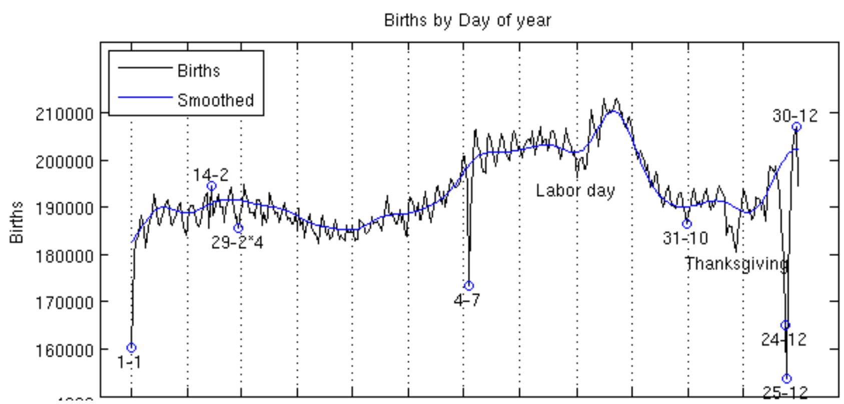
$$\begin{aligned} P(A_1, A_2) &= P(A_1)P(A_2|A_1) \\ &= (365/365) \times (364/365) \end{aligned}$$

$$\begin{aligned} P(A_1, A_2, A_3) &= P(A_1)P(A_2|A_1)P(A_3|A_1, A_2) \\ &= (365/365) \times (364/365) \times (363/365) \end{aligned}$$

$$\vdots$$

$$\begin{aligned} P(A_1, \ldots, A_n) &= \frac{365}{365}\frac{364}{365}\cdots\frac{365-(n-1)}{365} \\ &= \frac{365!}{(365-n)!\;365^n} \end{aligned}$$

Birthday Problem, cont.



The two armed bandit ...

If you know that one of the machines is good (pays out 50% of the time) while the other is bad (pays out 20% of the time) but not which machine is which, what should you do?

Picking a machine

Playing the left machine

What do we know?

- If it is the good machine:

$$P(W|G) = 0.5 \text{ and } P(L|G) = 0.5$$

- If it is the bad machine:

$$P(W|B) = 0.2 \text{ and } P(L|B) = 0.8$$

- We have no idea if we are playing the good or the bad machine:

$$P(G) = 0.5 \text{ and } P(B) = 0.5$$

What happens if we win?

You put a quarter in the machine on the left, spin, and you win!
What does this tell us about the machine on the left? In particular do we now know more about whether it is the good or bad machine?

$$\begin{aligned} P(G|W) &= \frac{P(W|G)\,P(G)}{P(W)} \\ &= \frac{P(W|G)P(G)}{P(W|G)P(G) + P(W|B)P(B)} \\ &= \frac{0.5 \times 0.5}{0.5 \times 0.5 + 0.2 \times 0.5} = \frac{5}{7} \end{aligned}$$

$$P(B|W) = 1 - P(G|W) = \frac{2}{7}$$

What happens if we win again?

After winning on on your first play you decide to put another quarter in the machine on the left again, and you win again. Now what do we know about the machine you played?

It turns out we can repeat the exact same calculation we did before but this time we just need to update our ‘belief‘ about which machine is good and which is bad.

Our new belief (prior) should reflect our previous experience (data):

$$P(G) = \frac{5}{7} = 0.714 \qquad P(B) = \frac{2}{7} = 0.286$$

Bayesian updates

Given our new belief we can quickly calculate a new probability that the machine on the left is the "good" machine:

$$
\begin{aligned}
P(G|W) &= \frac{P(W|G)\ P(G)}{P(W)} \\
&= \frac{P(W|G)P(G)}{P(W|G)P(G) + P(W|B)P(B)} \\
&= \frac{0.5 \times 0.714}{0.5 \times 0.714 + 0.2 \times 0.286} \\
&= 0.862
\end{aligned}
$$

$$
\begin{aligned}
P(B|W) &= 1 - P(G|W) \\
&= 0.138
\end{aligned}
$$

The long way

$$
\begin{aligned}
P(G|W_1, W_2) &= \frac{P(W_1, W_2|G)\ P(G)}{P(W_1, W_2)} \\
&= \frac{P(W_1|G)\ P(W_2|G)\ P(G)}{P(W_1)P(W_2|W_1)} \\
&= \frac{P(W_2|G)}{P(W_2|W_1)} \left[\frac{P(W_1|G)\ P(G)}{P(W_1)}\right] \\
&= \frac{P(W_2|G)}{P(W_2|W_1)} \Big[P(G|W_1)\Big] \\
&= \frac{P(W_2|G)}{P(W_2|G, W_1)P(G|W_1) + P(W_2|B, W_1)P(B|W_1)} \Big[P(G|W_1)\Big] \\
&= \frac{P(W_2|G)}{P(W_2|G)P(G|W_1) + P(W_2|B)P(B|W_1)} \Big[P(G|W_1)\Big] \\
&= \frac{1/2}{1/2 \times 5/7 + 1/5 \times 2/7}\ \frac{5}{7} = 25/29 = 0.862
\end{aligned}
$$

Why do we care?

The two-armed (multi-armed) bandit is a very useful model when it comes to clinical trials.

We are trying one or more treatments against a control and we want to know the efficacy of those treatments. This is much more complex in practice because not only do we not know which is better ($P(G)$ in our slot example) we also don't know how much better they are (also need to estimate $P(W|G)$).

Complex optimization problem where we must allocation a limited number of subjects to properly balance:

- Exploration - estimate the payoff of each treatment
- Exploitation - get the best outcome for the most patients

Back to House and Lupus

Last time we worked through a problem on the probability of a patient having lupus given they test positive. We were given

- $P(L) = 0.02$
- $P(+|L) = 0.99$
- $P(-|L^c) = 0.74$

From which we calculated that

$$
P(L|+) = \frac{P(L \cap +)}{P(+)} = \frac{P(+|L)P(L)}{P(+|L)P(L) + P(+|L^c)P(L^c)} = \frac{0.02 \times 0.99}{0.02 \times 0.99 + 0.98 \times 0.26} = 0.072
$$

If the patient gets a second test, how should our belief in the probability of having lupus, $P(L)$, change?

Let's Make a Deal. . .

Monty Hall Problem

You are offered a choice of three doors, there is a car behind one of the doors and there are goats behind the other two.

Monty Hall, Let's Make a Deal's original host, lets you choose one of the three doors.

Monty then opens one of the other two doors to reveal one of the goats.

You are then allowed to stay with your original choice or switch to the other door.

Which option should you choose?

(a) stay (b) switch (c) it does not matter

A Little History

First known formulation comes from a 1975 letter by Steve Selvin to the American Statistician.

Popularized in 1990 by Marilyn vos Savant in her "Ask Marilyn" column in Parade magazine.

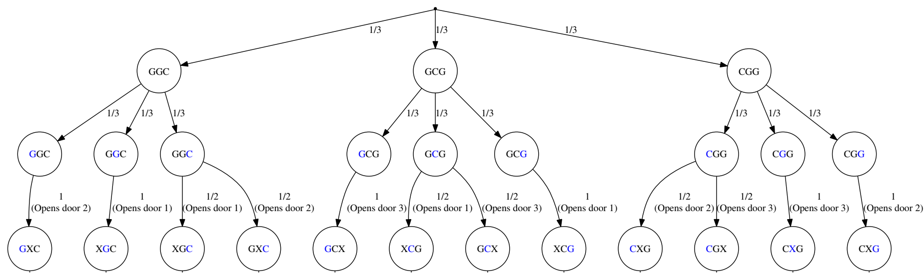
- vos Savant's solution claimed that the contestant should always switch
- About 10,000 (1,000 from Ph.D.s) letters contesting the solution
- vos Savant was right, easy to show with simulation

Moral of the story: trust the math not your intuition

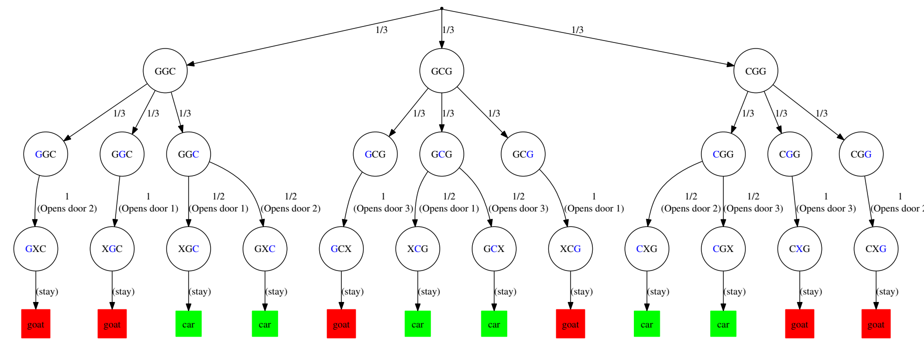
A slightly more entertaining variant of Monty Hall ...

`https://www.youtube.com/watch?v=tvODuUMLLgM&t=3m24s`

Monty Hall - The hard way

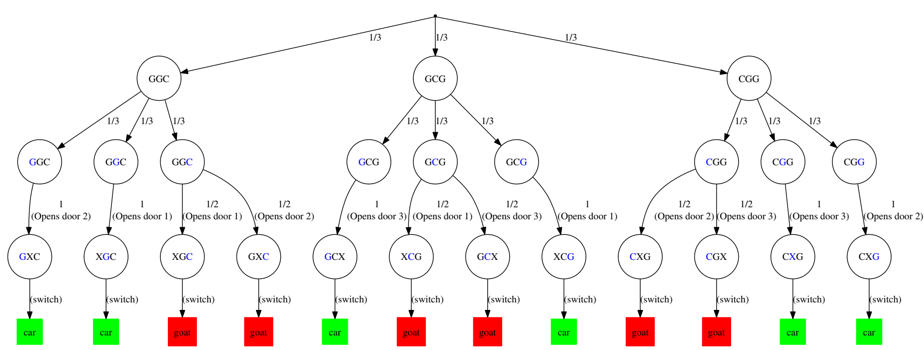


Monty Hall - The hard way - Stay



$$P(Car|Stay) = \frac{1}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{18} = 6/18 = 1/3$$

Monty Hall - The hard way - Switch



$$P(Car|Switch) = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = 6/9 = 2/3$$