Lecture 7 - Continuous Distributions (Normal)

Sta102 / BME 102

Colin Rundel

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Discrete Probability Distributions

A *discrete probability distribution* lists all possible events and the probabilities with which they occur.

Rules for discete probability distributions:

- The outcomes must be disjoint

$$P(X \cap Y) = 0 \text{ if } X \neq Y$$

- The probability of each outcome must be between 0 and 1

$$0 \leq P(X) \leq 1$$

- The sum of the probabilities of outcomes must total 1

$$\sum_{all\ x} P(X = x) = 1$$

Continuous Probability Distributions

A *continuous probability distribution* differs from a *discrete probability distribution* in several ways:

- The probability that a continuous RV will equal to any specific value is zero.
- As such, they cannot be expressed in tabular form or with a probability mass function.
- Instead, we describe its distribution via a *probability density function* - $f(x|\theta)$.

$$f(x) = \lim_{\epsilon \to 0} P(X \in \{x, x + \epsilon\})$$

- We can calculate probability for ranges of values (area under the curve given by the pdf).

$$P(a < X < b) = \int_a^b f(x|\theta)\ dx$$

Rules for density functions

A *probability density function* must have the following properties:

- The density must be positive everywhere,

$$f(x|\theta) \geq 0 \text{ for all } x \in (-\infty, \infty).$$

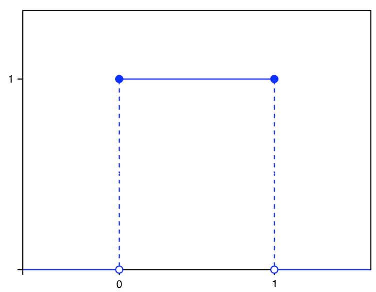
- The integral of the density (area under the curve) from $-\infty$ to ∞ must be 1,

$$\int_{-\infty}^{\infty} f(x|\theta)\ dx = 1.$$

Example - Uniform Distribution

If a random variable X has constant probability over a range $(0,1)$ then X has a standard uniform distribution, $X \sim \text{Unif}(0,1)$.

$$f(x) = \begin{cases} 1 & \text{for } x \in (0,1) \\ 0 & \text{otherwise} \end{cases}$$



Properties:

$E(X) = 1/2$

$Var(X) = 1/12$

$P(X = x) = 0$

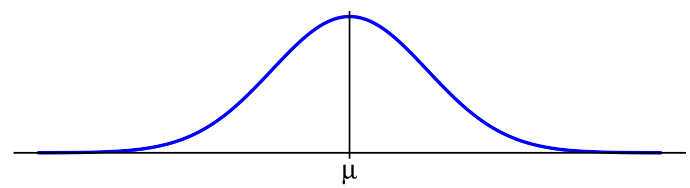
$P(a < X < b) = b - a$ for $a, b \in (0,1)$

Normal distribution

- Unimodal and symmetric, bell shaped curve
- Many variables are nearly normal, but almost none are exactly normal
- Density given by

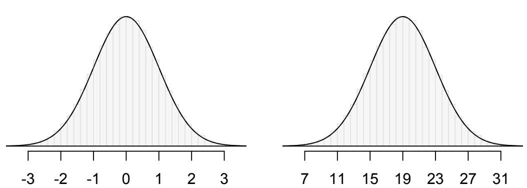
$$f(x|\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$$

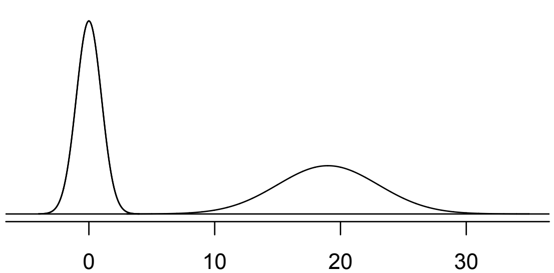
$X \sim N(\mu,\sigma)$ $\quad E(X) = \mu$ $\quad Var(X) = \sigma^2$ $\quad SD(X) = \sigma$



Normal distributions with different parameters

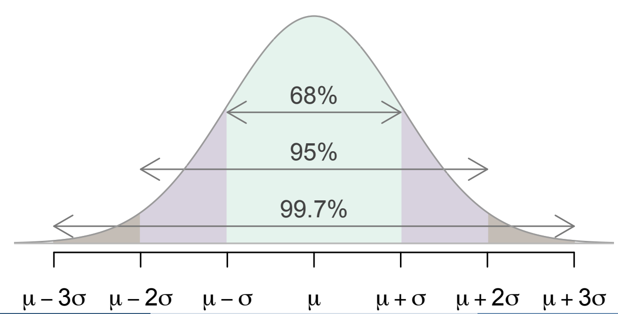
$N(\mu = 0, \sigma = 1)$ $\qquad$ $N(\mu = 19, \sigma = 4)$





68-95-99.7 Rule

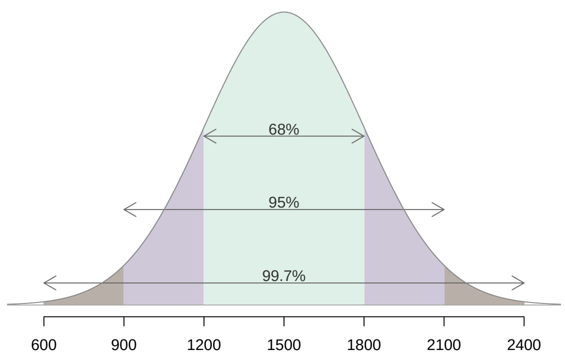
- For nearly normally distributed data,
  - about 68% falls within 1 SD of the mean,
  - about 95% falls within 2 SD of the mean,
  - about 99.7% falls within 3 SD of the mean.
- It is possible for observations to fall 4, 5, or more standard deviations away from the mean, but these occurrences are very rare if the data are nearly normal.



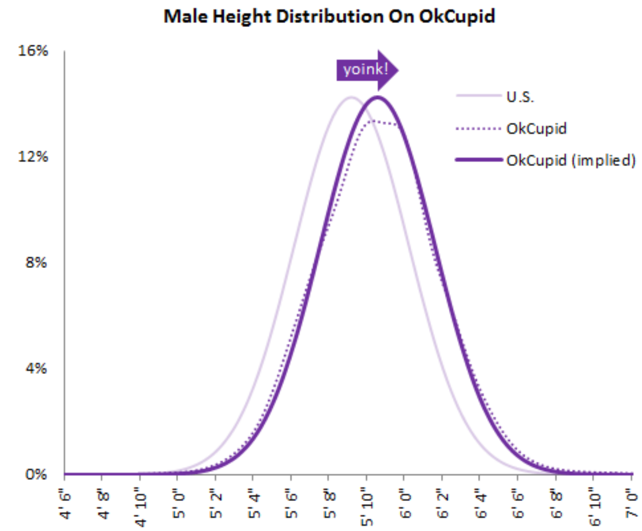
Describing variability using the 68-95-99.7 Rule

SAT scores are distributed nearly normally with mean 1500 and standard deviation 300.

- ~68% of students score between 1200 and 1800 on the SAT.
- ~95% of students score between 900 and 2100 on the SAT.
- ~99.7% of students score between 600 and 2400 on the SAT.



Heights of males



http://blog.okcupid.com/index.php/the-biggest-lies-in-online-dating/

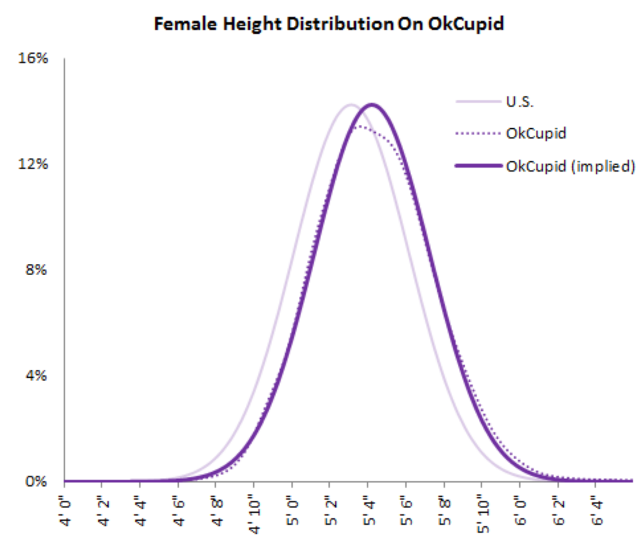
OkCupid's Take

"The male heights on OkCupid very nearly follow the expected normal distribution – except the whole thing is shifted to the right of where it should be. Almost universally guys like to add a couple inches."

"You can also see a more subtle vanity at work: starting at roughly 5' 8", the top of the dotted curve tilts even further rightward. This means that guys as they get closer to six feet round up a bit more than usual, stretching for that coveted psychological benchmark."

"When we looked into the data for women, we were surprised to see height exaggeration was just as widespread, though without the lurch towards a benchmark height."

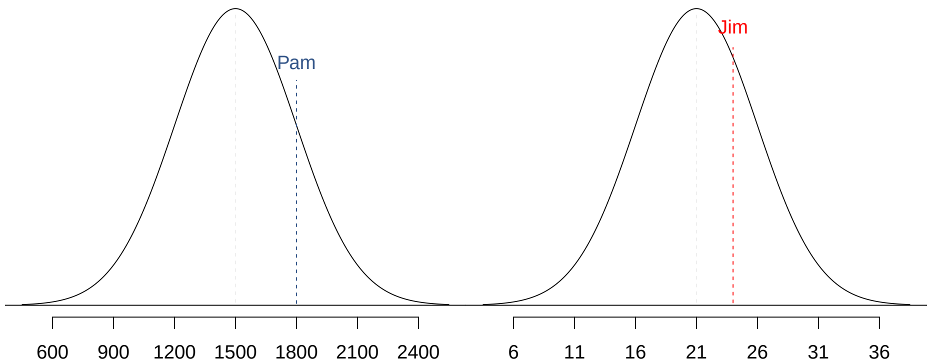
Heights of females



http://blog.okcupid.com/index.php/the-biggest-lies-in-online-dating/

Comparing SAT and ACT

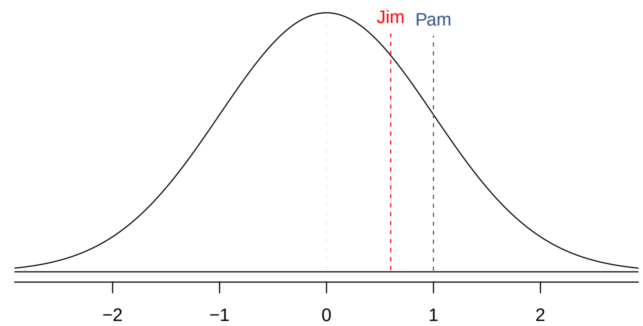
SAT scores are distributed nearly normally with mean 1500 and standard deviation 300. ACT scores are distributed nearly normally with mean 21 and standard deviation 5. A college admissions officer wants to determine which of the two applicants scored better on their standardized test with respect to the other test takers: Pam, who earned an 1800 on her SAT, or Jim, who scored a 24 on his ACT?



Standardizing

Since we cannot just compare these two raw scores, we instead compare how many standard deviations above or below the mean each observation is.

- Pam's score is $\frac{1800-1500}{300} = 1$ standard deviation above the mean.
- Jim's score is $\frac{24-21}{5} = 0.6$ standard deviations above the mean.



Standardizing with Z scores (cont.)

Z / *standardized* / *normalized* scores

- A measure of the number of standard deviations the data falls above or below the mean.

$$Z = \frac{observation - mean}{SD}$$

- Z scores are defined for distributions of any shape, but only when the distribution is normal can we use Z scores to calculate probabilities.
- Observations that are more than 2 SD away from the mean ($|Z| > 2$) are typically considered unusual.

Z distribution

Another reason we use Z scores is if the distribution of X is nearly normal then the Z scores of X will have a Z distribution.

- Z distribution is a special case of the normal distribution where $\mu = 0$ and $\sigma = 1$ (unit normal distribution)
- Linear transformations of normally distributed random variable will also be normally distributed. Hence, if

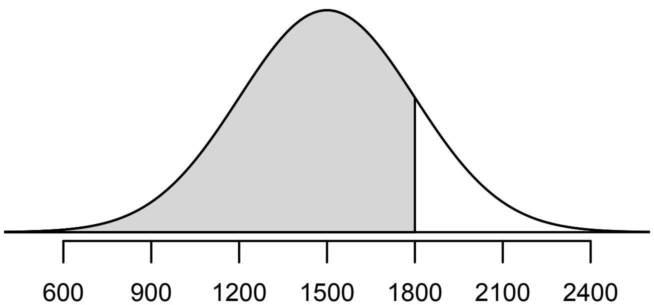
$$Z = \frac{X-\mu}{\sigma}, \text{ where } X \sim N(\mu, \sigma)$$

$$E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = E(X/\sigma) - \mu/\sigma = 0$$

$$Var(Z) = Var\left(\frac{X-\mu}{\sigma}\right) = Var(X/\sigma) = \frac{1}{\sigma^2}Var(X) = 1$$

Percentiles

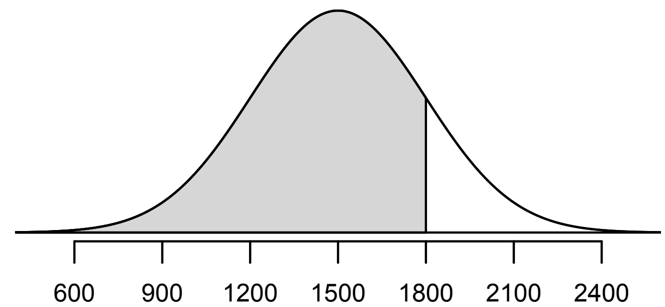
- *Percentile* is the percentage of observations that fall below a given data point.
- Graphically, the percentile is the area below the probability distribution curve to the left of then observation.



Example - SAT

Approximately what percent of students score below 1800 on the SAT?

$$\mu = 1500, \quad \sigma = 200$$



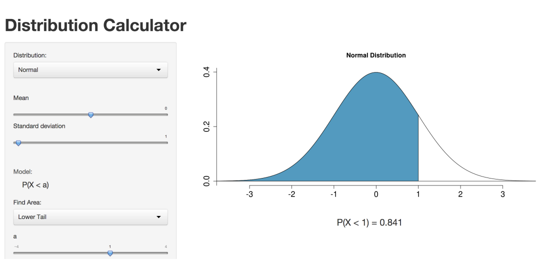
Calculating percentiles

There are many ways to compute percentiles/areas under the curve:

- R:

  pnorm(1800, mean = 1500, sd = 300))

- Applet:



http://spark.rstudio.com/minebocek/dist_calc/

- Calculus:

$$P(X \leq 1800) = \Phi(1800) = \int_{-\infty}^{1800} \frac{1}{300\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x-1500}{300}\right)^2\right] dx$$

Calculating percentiles, cont.

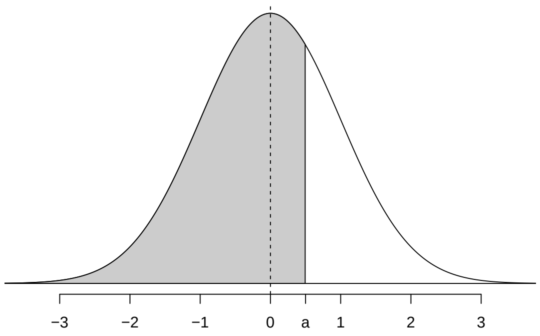
Z Table:

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	*0.8413*	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015

Calculating left tail probabilities

The area under the unit normal curve from $-\infty$ to a is given by

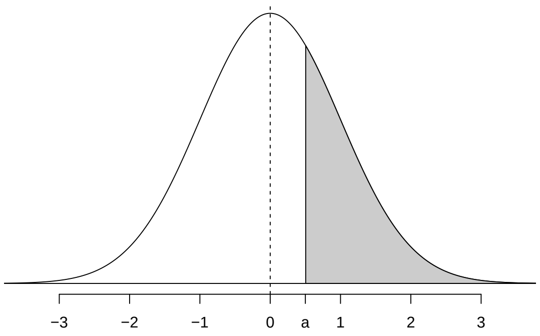
$$P(Z \le a) = \Phi(a)$$



Calculating right tail probabilities

The area under the unit normal curve from a to ∞ is given by

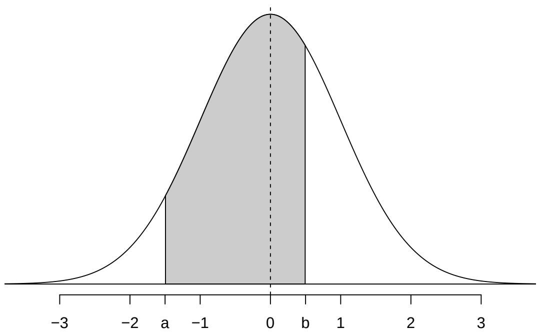
$$P(Z \ge a) = 1 - \Phi(a)$$



Calculating middle probabilities

The area under the unit normal curve from a to b where $a \le b$ is given by

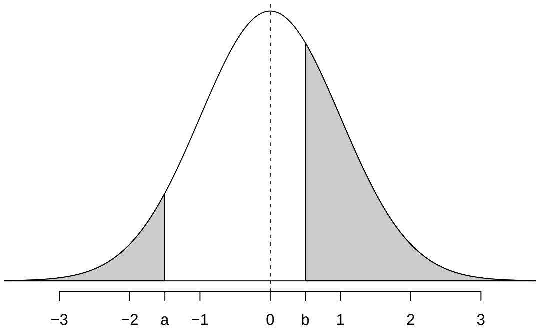
$$P(a \le Z \le b) = \Phi(b) - \Phi(a)$$



Calculating two tail probabilities

The area under the unit normal curve outside of a to b where $a \le b$ is given by

$$P(a \ge Z \text{ or } Z \ge b) = \Phi(a) + (1 - \Phi(b)) = 1 - (\Phi(b) - \Phi(a))$$



Φ Practice

How would you calculate the following probability?

$$P(Z < -1)$$

How would you calculate the following probability?

$$P(Z > 2.22)$$

How would you calculate the following probability?

$$P(-1.53 \leq Z \leq 2.75)$$

How would you calculate the following probability?

$$P(Z \leq 0.75 \text{ or } Z \geq 1.43)$$

Probabilities for non-Unit Normal Distributions

Everything we just discussed on the previous 4 slides applies only to the unit normal distribution, but this doesn't come up very often in problems.

Let X be a normally distributed random variable with mean μ and variance σ^2 then we define the random variable Z such that

$$Z = \left(\frac{X-\mu}{\sigma}\right) \sim N(0,1)$$

$$P(a \leq X \leq b) = P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

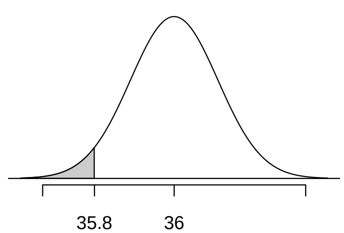
Example - Dosage

At a pharmaceutical factory the amount of the active ingredient which is added to each pill is supposed to be 36 mg. The amount of the active ingredient added follows a nearly normal distribution with a standard deviation of 0.11 mg. Once every 30 minutes a pill is selected from the production line, and its composition is measured precisely. If the amount of the active ingredient in the pill is below 35.8 mg or above 36.2 mg, then that production run of pills fails the quality control inspection. What percent of production runs will fail for having too little active ingredient (less than 35.8 mg)?

Let X = amount of active ingredient in a pill, $X \sim \mathrm{N}(\mu = 36, \sigma = 0.11)$

$$Z = \frac{x-\mu}{\sigma} = \frac{35.8 - 36}{0.11} = -1.82$$

$$P(X < 35.8) = P(Z < -1.82) = ?$$



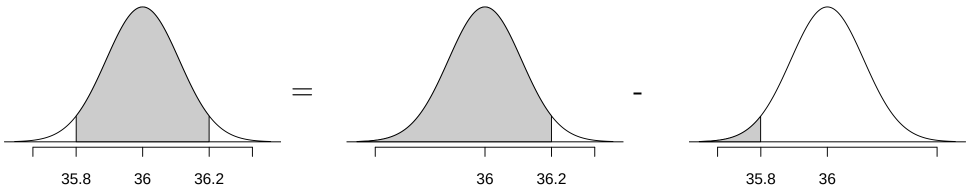
Finding the exact probability

Second decimal place of Z										
0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00	Z
0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019	−2.9
0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026	−2.8
0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035	−2.7
0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047	−2.6
0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062	−2.5
0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082	−2.4
0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107	−2.3
0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139	−2.2
0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179	−2.1
0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228	−2.0
0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287	−1.9
0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359	−1.8
0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446	−1.7
0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548	−1.6
0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668	−1.5

Example - Dosage pt. 2

At the same pharmaceutical factory ($\mu = 36$ oz and $\sigma = 0.11$ oz). What percent of production runs pass the quality control inspection (between 35.8 and 36.2 mg of active ingredient in the tested pill)?

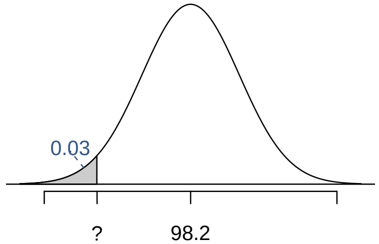
$$P(35.8 < X < 36.2) = ?$$



$$Z_{35.8} = \frac{35.8 - 36}{0.11} = -1.82$$
$$Z_{36.2} = \frac{36.2 - 36}{0.11} = 1.82$$
$$P(35.8 < X < 36.2) = P(-1.82 < Z < 1.82)$$
$$= 0.9656 - 0.0344 = 0.9312$$

Example - Body Temperature

Body temperatures of healthy humans are distributed nearly normally with mean 98.2°F and standard deviation 0.73°F. What is the cutoff for the lowest 3% of human body temperatures? (Mackowiak, Wasserman, and Levine 1992)



0.09	0.08	0.07	0.06	0.05	Z
0.0233	0.0239	0.0244	0.0250	0.0256	-1.9
0.0294	*0.0301*	0.0307	0.0314	0.0322	-1.8
0.0367	0.0375	0.0384	0.0392	0.0401	-1.7

$$P(X < x) = 0.03 \rightarrow P(Z < -1.88) = 0.03$$
$$Z = \frac{obs - mean}{SD} \rightarrow \frac{x - 98.2}{0.73} = -1.88$$
$$x = (-1.88 \times 0.73) + 98.2 = 96.8$$

Example - Body Temperature pt. 2

What is the cutoff for the highest 10% of human body temperatures?