

Lecture 18 - χ^2 Tests

Sta102 / BME102

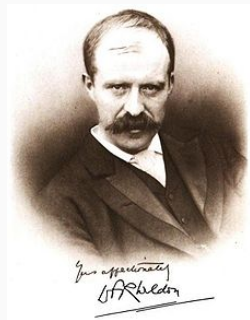
April 6, 2016

Colin Rundel

χ^2 test of GOF

Weldon's dice

- Walter Frank Raphael Weldon (1860 - 1906), was an English evolutionary biologist and a founder of biometry. He was the joint founding editor of *Biometrika*, with Francis Galton and Karl Pearson.
- In 1894, he rolled 12 dice 26,306 times, and recorded the number of 5s or 6s (which he considered to be a success).
- 5s or 6s occurred more often than expected - Pearson hypothesized that since inexpensive dice have hollowed-out pips the face with 6 pips is lighter than its opposing face, which has only 1 pip.



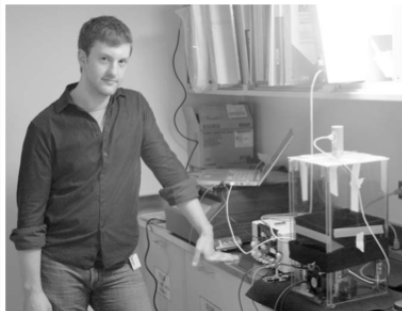
Weldon's Data

| Number of Successes | Observed Frequency | Theoretical Frequency, $p = 1/3$ | Deviation |
|---------------------|--------------------|-------------------------------------|-----------|
| 0 | 185 | 203 | -18 |
| 1 | 1149 | 1216 | -67 |
| 2 | 3265 | 3345 | -80 |
| 3 | 5475 | 5576 | -101 |
| 4 | 6114 | 6273 | -159 |
| 5 | 5194 | 5018 | 176 |
| 6 | 3067 | 2927 | 140 |
| 7 | 1331 | 1255 | 76 |
| 8 | 403 | 392 | 11 |
| 9 | 105 | 87 | 18 |
| 10 | 14 | 13 | 1 |
| 11 | 4 | 1 | 3 |
| 12 | 0 | 0 | 0 |
| Total | 26,306 | 26,306 | |

Labby's dice

In 2009, Zacariah Labby (U of Chicago), repeated Weldon's experiment using a homemade dice-throwing, pip counting machine.

<http://www.youtube.com/watch?v=95EErdouO2w>

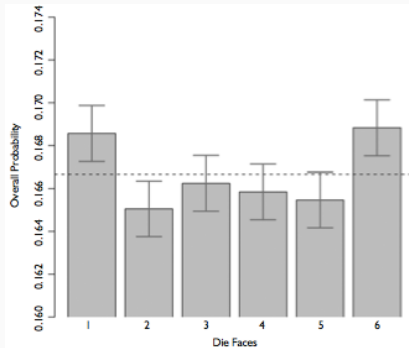


- Able to recreate Weldon's experiment in about six days (with complete data).
- Further reading: <http://news.uchicago.edu/static/newsengine/pdf/labby09dice.pdf>

Labby's dice (cont.)

Labby did not actually observe the same phenomenon that Weldon observed (higher frequency of 5s and 6s).

Automation allowed Labby to collect more data than Weldon did in 1894, instead of recording “successes” and “failures”, Labby recorded the individual number of pips on each die.



Summarizing Labby's results

Labby rolled 12 dice 26,306 times. If each side is equally likely to come up, how many 1s, 2s, \dots , 6s would he expect to have observed?

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$$\frac{12 \times 26,306}{6} = 52612$$

| Outcome | Observed | Expected |
|---------|----------|----------|
| 1 | 53,222 | 52,612 |
| 2 | 52,118 | 52,612 |
| 3 | 52,465 | 52,612 |
| 4 | 52,338 | 52,612 |
| 5 | 52,244 | 52,612 |
| 6 | 53,285 | 52,612 |
| Total | 315,672 | 315,672 |

Setting the hypotheses

Do these data provide convincing evidence to suggest an inconsistency between the observed and expected counts?

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H_0 : There is no inconsistency between the observed and the expected counts. *The observed counts follow the same distribution as the expected counts.*

H_A : There is an inconsistency between the observed and the expected counts. *The observed counts do not follow the same distribution as the expected counts.* (There is a bias in which side comes up on the roll of a die)

Evaluating the hypotheses

To evaluate these hypotheses, we quantify how different the observed counts are from the expected counts.

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This is called a *goodness of fit* test since we're evaluating how well the observed data fit the expected distribution.

Anatomy of a test statistic

The general form of the test statistics we've seen this far is

$$\frac{\text{point estimate} - \text{null value}}{\text{SE of point estimate}}$$

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2. standardizing that difference using the standard error of the point estimate.

These two ideas will help in the construction of an appropriate test statistic for count data.

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The χ^2 statistic is defined to be

$$\chi^2 = \sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i}$$

where k = total number of cells/categories

Calculating the χ^2 statistic

| Outcome | Observed | Expected | $\frac{(O-E)^2}{E}$ |
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| 1 | 53,222 | 52,612 | $\frac{(53,222-52,612)^2}{52,612} = 7.07$ |

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| Total | 315,672 | 315,672 | 24.73 |

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Where have we seen this before?

Conditions for the χ^2 test

1. *Independence*: Each case that contributes a count to the table must be independent of all the other cases in the table.

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1. *Independence*: Each case that contributes a count to the table must be independent of all the other cases in the table.
2. *Sample size*: Each particular scenario (i.e. cell) must have at least 5 *expected* cases.

Failing to check conditions may unintentionally effect the test's error rates.

The χ^2 distribution

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So far we've seen three other continuous distributions:

- Normal - unimodal and symmetric with two parameters: μ (center) and σ^2 (spread)
- T - unimodal and symmetric with one parameter: df (spread, kurtosis)
- T - unimodal and non-symmetric with two parameters: df_1, df_2

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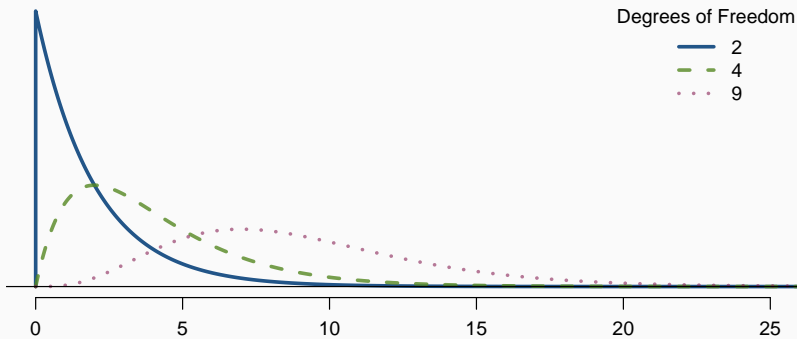
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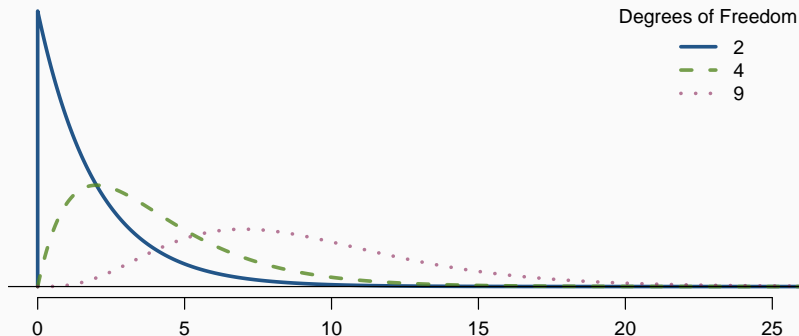
- If we have $Z_1, \dots, Z_n \sim N(0, 1)$ then the quantity,

$$Z_1^2 + Z_2^2 + \dots + Z_n^2 \stackrel{iid}{\sim} \chi_{df=n}^2$$

The χ^2 distribution (cont.)



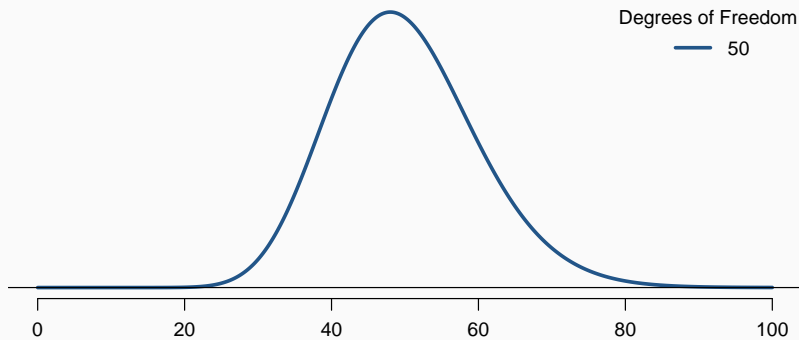
The χ^2 distribution (cont.)



As the df increases:

- the center of the χ^2 distribution increases
- the variability of the χ^2 distribution increases

The χ^2 distribution (cont.)



Also, for large df the χ^2 distribution converges to the normal distribution with

$$\mathcal{N}(\mu = df, \sigma^2 = 2 df).$$

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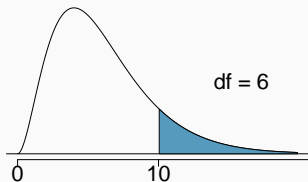
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- For this we can use technology, or a χ^2 table.
- This table is similar to the t table, it provides upper tail probabilities.

| Upper tail | | 0.3 | 0.2 | 0.1 | 0.05 | 0.02 | 0.01 | 0.005 | 0.001 |
|------------|-----|------|------|-------|-------|-------|-------|-------|-------|
| df | 1 | 1.07 | 1.64 | 2.71 | 3.84 | 5.41 | 6.63 | 7.88 | 10.83 |
| | 2 | 2.41 | 3.22 | 4.61 | 5.99 | 7.82 | 9.21 | 10.60 | 13.82 |
| | 3 | 3.66 | 4.64 | 6.25 | 7.81 | 9.84 | 11.34 | 12.84 | 16.27 |
| | 4 | 4.88 | 5.99 | 7.78 | 9.49 | 11.67 | 13.28 | 14.86 | 18.47 |
| | 5 | 6.06 | 7.29 | 9.24 | 11.07 | 13.39 | 15.09 | 16.75 | 20.52 |
| | 6 | 7.23 | 8.56 | 10.64 | 12.59 | 15.03 | 16.81 | 18.55 | 22.46 |
| | 7 | 8.38 | 9.80 | 12.02 | 14.07 | 16.62 | 18.48 | 20.28 | 24.32 |
| | ... | | | | | | | | |

Finding areas under the χ^2 curve (cont.)

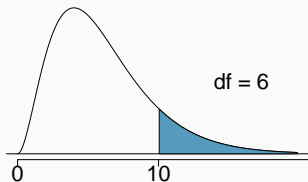
Estimate the shaded area under the χ^2 curve with $df = 6$.



| Upper tail | 0.3 | 0.2 | 0.1 | 0.05 | 0.02 | 0.01 | 0.005 | 0.001 |
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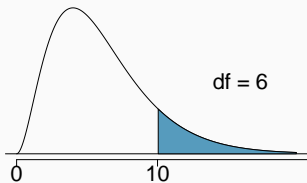
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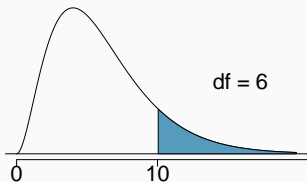
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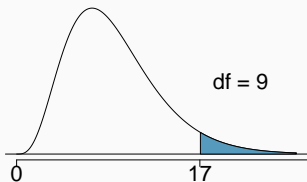


$P(\chi_{df=6}^2 > 10)$
is between 0.1 and 0.2

| Upper tail | 0.3 | 0.2 | 0.1 | 0.05 | 0.02 | 0.01 | 0.005 | 0.001 |
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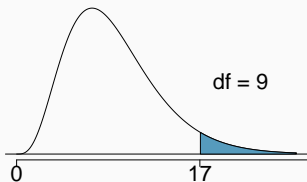
Estimate the shaded area (above 17) under the χ^2 curve with $df = 9$.



| Upper tail | 0.3 | 0.2 | 0.1 | 0.05 | 0.02 | 0.01 | 0.005 | 0.001 |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|
| df 7 | 8.38 | 9.80 | 12.02 | 14.07 | 16.62 | 18.48 | 20.28 | 24.32 |
| 8 | 9.52 | 11.03 | 13.36 | 15.51 | 18.17 | 20.09 | 21.95 | 26.12 |
| 9 | 10.66 | 12.24 | 14.68 | 16.92 | 19.68 | 21.67 | 23.59 | 27.88 |
| 10 | 11.78 | 13.44 | 15.99 | 18.31 | 21.16 | 23.21 | 25.19 | 29.59 |
| 11 | 12.90 | 14.63 | 17.28 | 19.68 | 22.62 | 24.72 | 26.76 | 31.26 |

Finding areas under the χ^2 curve (cont.)

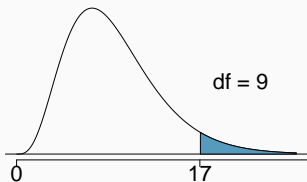
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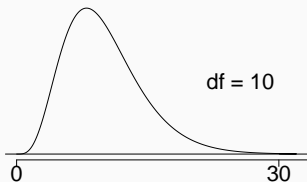


$P(\chi_{df=9}^2 > 17)$
is between 0.02 and 0.05

| Upper tail | 0.3 | 0.2 | 0.1 | 0.05 | 0.02 | 0.01 | 0.005 | 0.001 |
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Finding areas under the χ^2 curve (one more)

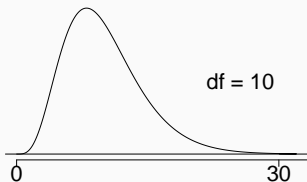
Estimate the shaded area (above 30) under the χ^2 curve with $df = 10$.



| Upper tail | 0.3 | 0.2 | 0.1 | 0.05 | 0.02 | 0.01 | 0.005 | 0.001 |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|
| df 7 | 8.38 | 9.80 | 12.02 | 14.07 | 16.62 | 18.48 | 20.28 | 24.32 |
| 8 | 9.52 | 11.03 | 13.36 | 15.51 | 18.17 | 20.09 | 21.95 | 26.12 |
| 9 | 10.66 | 12.24 | 14.68 | 16.92 | 19.68 | 21.67 | 23.59 | 27.88 |
| 10 | 11.78 | 13.44 | 15.99 | 18.31 | 21.16 | 23.21 | 25.19 | 29.59 |
| 11 | 12.90 | 14.63 | 17.28 | 19.68 | 22.62 | 24.72 | 26.76 | 31.26 |

Finding areas under the χ^2 curve (one more)

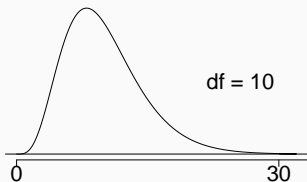
Estimate the shaded area (above 30) under the χ^2 curve with $df = 10$.



| Upper tail | 0.3 | 0.2 | 0.1 | 0.05 | 0.02 | 0.01 | 0.005 | 0.001 | |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|---|
| df 7 | 8.38 | 9.80 | 12.02 | 14.07 | 16.62 | 18.48 | 20.28 | 24.32 | → |
| 8 | 9.52 | 11.03 | 13.36 | 15.51 | 18.17 | 20.09 | 21.95 | 26.12 | |
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Finding areas under the χ^2 curve (one more)

Estimate the shaded area (above 30) under the χ^2 curve with $df = 10$.



$P(\chi_{df=10}^2 > 30)$
is less than 0.001

| Upper tail | 0.3 | 0.2 | 0.1 | 0.05 | 0.02 | 0.01 | 0.005 | 0.001 | → |
|------------|-------|-------|-------|-------|-------|-------|-------|-------|---|
| df 7 | 8.38 | 9.80 | 12.02 | 14.07 | 16.62 | 18.48 | 20.28 | 24.32 | |
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pchisq(q = 30, df = 10, lower.tail = FALSE)
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pchisq(q = 30, df = 10, lower.tail = FALSE)
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- Using a web applet -

https://gallery.shinyapps.io/dist_calc/

Back to Labby's dice

Our original question was: Does Labby's data provide convincing evidence to suggest an inconsistency between the observed and expected counts (assuming the dice are fair)?

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All we need is the df and we can calculate the tail area (the p-value) and make a decision on the hypotheses.

Degrees of freedom for a goodness of fit test

When conducting a goodness of fit test to evaluate how well the observed data follow an expected distribution, the degrees of freedom are calculated as the number of cells (k) minus 1.

$$df = k - 1$$

Degrees of freedom for a goodness of fit test

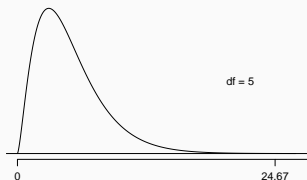
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For the dice, $k = 6$ therefore

$$df = 6 - 1 = 5$$

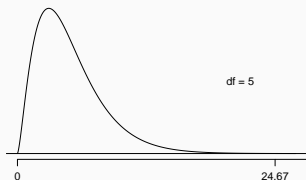
Finding a p-value for a χ^2 test



$$p\text{-value} = P(\chi_{df=5}^2 > 24.67)$$

| Upper tail | 0.3 | 0.2 | 0.1 | 0.05 | 0.02 | 0.01 | 0.005 | 0.001 |
|------------|------|------|------|-------|-------|-------|-------|-------|
| df 1 | 1.07 | 1.64 | 2.71 | 3.84 | 5.41 | 6.63 | 7.88 | 10.83 |
| 2 | 2.41 | 3.22 | 4.61 | 5.99 | 7.82 | 9.21 | 10.60 | 13.82 |
| 3 | 3.66 | 4.64 | 6.25 | 7.81 | 9.84 | 11.34 | 12.84 | 16.27 |
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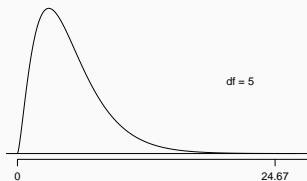
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|------------|---|------|------|------|-------|-------|-------|-------|-------|---|
| df | 1 | 1.07 | 1.64 | 2.71 | 3.84 | 5.41 | 6.63 | 7.88 | 10.83 | |
| | 2 | 2.41 | 3.22 | 4.61 | 5.99 | 7.82 | 9.21 | 10.60 | 13.82 | |
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Finding a p-value for a χ^2 test



$$p\text{-value} = P(\chi_{df=5}^2 > 24.67) < 0.001$$

| Upper tail | | 0.3 | 0.2 | 0.1 | 0.05 | 0.02 | 0.01 | 0.005 | 0.001 | → |
|------------|---|------|------|------|-------|-------|-------|-------|-------|---|
| df | 1 | 1.07 | 1.64 | 2.71 | 3.84 | 5.41 | 6.63 | 7.88 | 10.83 | |
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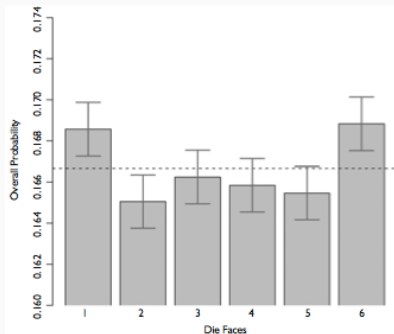
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Turns out...

Pearson's claim that 5s and 6s appear more often due to the carved-out pips is not supported by these data.

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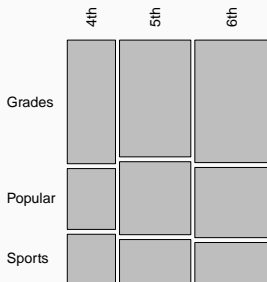
Labby found that the 1-6 axis is consistently shorter than the other two (2-5 and 3-4), the faces with one and six pips are larger than the other faces.

χ^2 test of independence

Popular kids

Students in grades 4-6 were asked whether good grades, athletic ability, or popularity was most important to them. A two-way table separating the students by grade and by choice of most important factor is shown below. Do these data provide evidence to suggest that goals vary by grade?

| | Grades | Popular | Sports |
|-----------------|--------|---------|--------|
| 4 th | 63 | 31 | 25 |
| 5 th | 88 | 55 | 33 |
| 6 th | 96 | 55 | 32 |



χ^2 test of independence

Our hypotheses are:

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Conditions for the χ^2 test of independence

- *Independence*: Each case that contributes a count to the table must be independent of all the other cases in the table.
- *Sample size*: Each cell must have at least 5 *expected* counts.

χ^2 test of independence

The test statistic is calculated using

$$\chi_{df}^2 = \sum_{i=1}^k \frac{(O - E)^2}{E}$$

where k is the number of cells and $df = (R - 1) \times (C - 1)$.

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The p-value is the area under the χ^2 distribution curve to the right of the calculated test statistic,

$$\text{p-value} = P \left(\chi_{df}^2 > \sum_{i=1}^k \frac{(O - E)^2}{E} \right).$$

χ^2 test of independence (cont.)

Expected counts in two-way tables:

$$\text{Expected Counts} = \frac{(\text{row total}) \times (\text{column total})}{\text{table total}}$$

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| 4 th | 63 | 31 | 25 | 119 |
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| Total | 247 | 141 | 90 | 478 |

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$$E_{\text{row 1, col 2}} = \frac{119 \times 141}{478} = 35$$

Expected counts in two-way tables

What is the expected count for the highlighted cell?

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$$E_{5,Pop} = \frac{176 \times 141}{478} = 52$$

more 5th graders than expected have a goal of being popular

Calculating the test statistic in two-way tables

Expected counts are shown in (blue) next to the observed counts.

| | Grades | Popular | Sports | Total |
|-----------------|---------|---------|---------|-------|
| 4 th | 63 (61) | 31 (35) | 25 (23) | 119 |
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$$\chi^2 = \sum \frac{(63 - 61)^2}{61} + \frac{(31 - 35)^2}{35} + \dots + \frac{(32 - 34)^2}{34} = 1.3121$$

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$$\chi^2 = \sum \frac{(63 - 61)^2}{61} + \frac{(31 - 35)^2}{35} + \dots + \frac{(32 - 34)^2}{34} = 1.3121$$
$$df = (R - 1) \times (C - 1) = (3 - 1) \times (3 - 1) = 2 \times 2 = 4$$

Calculating the p-value

What is the correct p-value for this hypothesis test

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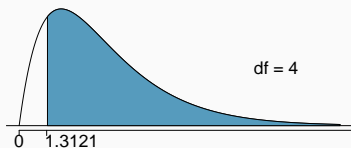
| Upper tail | | 0.3 | 0.2 | 0.1 | 0.05 | 0.02 | 0.01 | 0.005 | 0.001 |
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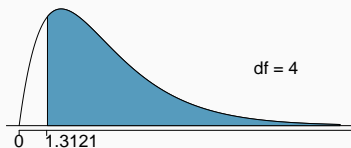


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$P(\chi_{df=4}^2 > 1.3121)$ is more than 0.3
P-value > 0.3

Conclusion

Do these data provide evidence to suggest that goals vary by grade?

H_0 : Grade and goals are independent. Goals do not vary by grade.

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Since p -value is large, we fail to reject H_0 . The data do not provide convincing evidence that grade and goals are dependent. It doesn't appear that goals vary by grade.

Summary

Summary - χ^2 test of goodness of fit

- Data:
 - y - categorical variable w/ 3 or more levels,
 - x - none.
- Hypotheses:
 - H_0 - data follow the given distribution,
 - H_A - data do not follow the given distribution.
- Conditions:
 - Independent observations, all $E_i \geq 5$.
- Test statistic:

$$\chi^2 = \sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i}, \quad df = k - 1$$

Summary - χ^2 test of independence

- Data:
 - y - categorical variable w/ 2 or more levels,
 - x - categorical variable w/ 2 or more levels.
- Hypotheses:
 - H_0 - x and y are independent,
 - H_A - x and y are dependent.
- Conditions:
 - Independent observations, all $E_{i,j} \geq 5$.

- Test statistic:

$$\chi^2 = \sum_{i=1}^m \sum_{j=1}^n \frac{(O_{i,j} - E_{i,j})^2}{E_{i,j}}, \quad df = (m - 1)(n - 1)$$

$$E_{i,j} = \frac{N_{i,\cdot} \times N_{\cdot,j}}{N_{\cdot,\cdot}}$$