# Lecture 6 - Properties of Random Variables

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# Random Variables

# Mean and variance of a discrete RVs

Last time we were introduced to some definitions for calculating the expected value (center) and variance (spead) for discrete random variables (distributions).

- Expected Value

$$E(X) = \sum_x x \cdot P(X = x)$$

- Variance

$$\begin{aligned} Var(X) &= E\Big(\big(X - E(X)\big)^2\Big) = E(X^2) - E(X)^2 \\ &= \sum_x \big(x - E(X)\big)^2 \cdot P(X = x) \end{aligned}$$

- Standard Deviation

$$SD(X) = \sqrt{Var(X)}$$

# Bernoulli Random Variable

A random variable for modeling binary events,

- Two possible outcomes:
  - Success - value 1
  - Failure - value 0
- Single parameter $p$, probability of a success

Let $X \sim \text{Bern}(p)$ then

$$P(X = x) = \begin{cases} 1 - p & \text{if } x = 0, \\ p & \text{if } x = 1 \end{cases}$$

$$\begin{aligned} E(X) &= p \\ Var(X) &= p(1-p) \end{aligned}$$

# Geometric Random Variable

A random variable that models the number of (identical) Bernoulli trials needed to obtain the first success.

- Infinite outcomes - $\{1, 2, 3, \ldots, \infty\}$
- Single parameter $p$, probability of a success for each trial

Let $X \sim \text{Geo}(p)$ then

$$P(X = x) = p(1-p)^{x-1}$$

$$E(X) = 1/p$$
$$Var(X) = \frac{1-p}{p^2}$$

# Binomial Random Variables

A random variable that models the *number of successes* in a *fixed number* of *independent Bernoulli trials*.

- Discrete outcomes - $\{0, 1, 2, 3, \ldots, n\}$
- Two parameters:
  - $p$ - probability of a success for each trial
  - $n$ - number of trials

Let $X \sim \text{Binom}(n, p)$ then

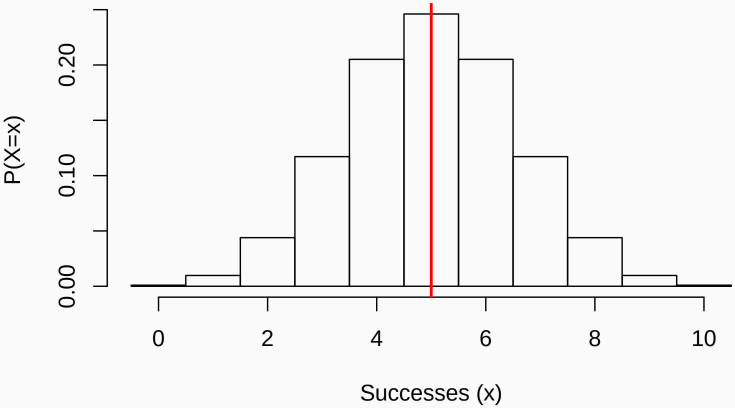
$$P(X = x|n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$E(X) = np$$

$$Var(X) = np(1-p)$$

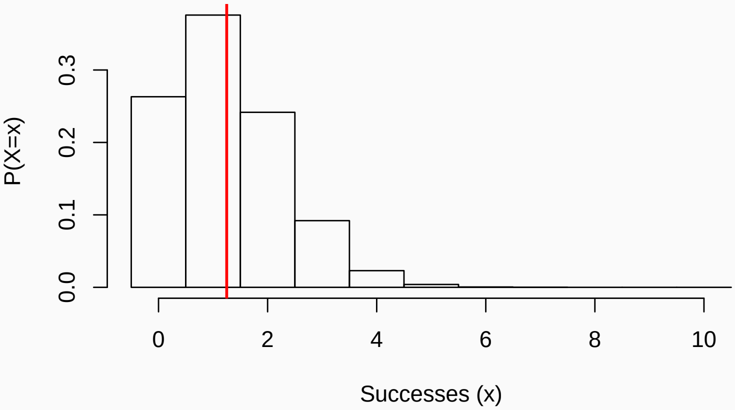
# Binomial RVs - Example 1

Let $X \sim \text{Binom}(n = 10, p = 1/2)$, $\quad E(X) = 10 \times 1/2 = 5$



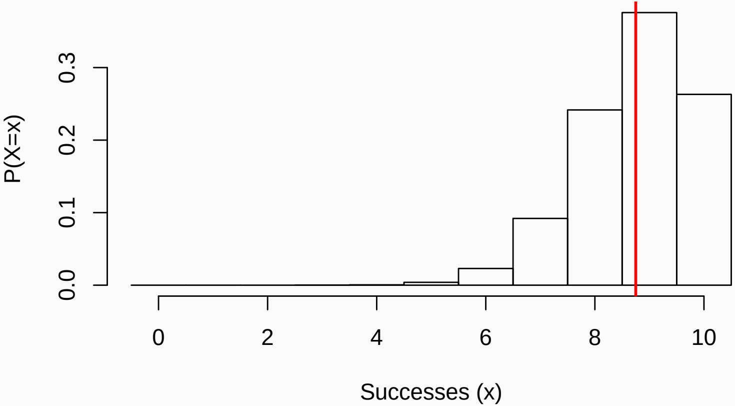
# Binomial RVs - Example 2

Let $X \sim \text{Binom}(n = 10, p = 1/8)$
$E(X) = 10 \times 1/8 = 1.25$



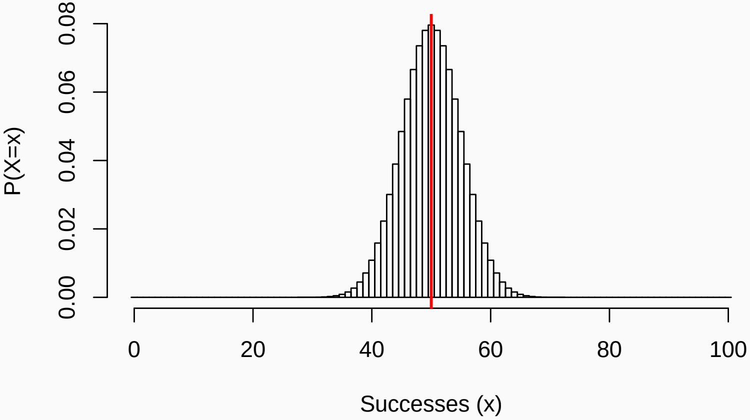
# Binomial RVs - Example 3

Let $X \sim \text{Binom}(n = 10, p = 7/8)$
$E(X) = 10 \times 7/8 = 8.75$



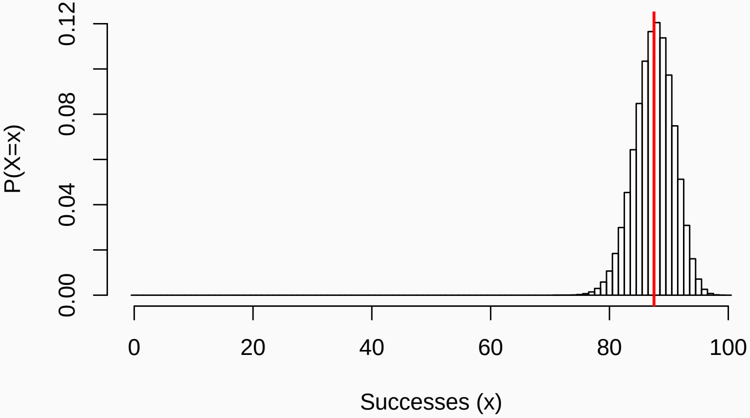
# Binomial RVs - Example 4

Let $X \sim \text{Binom}(n = 100, p = 4/8)$
$E(X) = 100 \times 4/8 = 50$



# Binomial RVs - Example 5

Let $X \sim \text{Binom}(n = 100, p = 7/8)$
$E(X) = 100 \times 7/8 = 70$



# Expected Value

# Properties of Expected Value

- **Constant** - $E(c) = c$ if $c$ is constant
- **Constant Multiplication** - $E(cX) = cE(X)$
- **Constant Addition** - $E(X + c) = E(X) + c$
- **Addition** - $E(X + Y) = E(X) + E(Y)$
- **Subtraction** - $E(X - Y) = E(X) - E(Y)$
- **Multiplication** - $E(XY) = E(X)E(Y)$ if $X$ and $Y$ are independent.

# Constant

Assume that $C$ is a random variable where $P(C = c) = 1$ then,

## Constant Multiplication

Assume $X$ is a discrete random variable and $c$ is some constant value then,

# Constant Addition

Assume $X$ is a discrete random variable and $c$ is some constant value then,

# Addition$^\star$

Assume $X$ and $Y$ are independent discrete random variables then,

# Variance

# Properties of Variance

- **Constant** - $Var(c) = 0$ if $c$ is constant
- **Constant Multiplication** - $Var(cX) = c^2\ Var(x)$
- **Constant Addition** - $Var(X + c) = Var(X)$
- **Addition** - $Var(X + Y) = Var(X) + Var(Y)$ if $X$ and $Y$ are independent.
- **Subtraction** - $Var(X - Y) = Var(X) + Var(Y)$ if $X$ and $Y$ are independent.

# Constant

Assume that $C$ is a random variable where $P(C = c) = 1$ then,

# Constant Multiplication

Assume $X$ is a discrete random variable and $c$ is some constant value then,

# Constant Addition

Assume $X$ is a discrete random variable and $c$ is some constant value then,

# Examples

# Example - Coffee

The average price of a small cup of coffee to go is \$1.40, with a standard deviation of 30¢. An 8.5% tax is added if you take your coffee to stay. Assume that each time you get a coffee to stay you also tip 50¢. What is the mean, variance, and standard deviation of the amount you spend on coffee when to take it to stay?

Let $X$ represent the amount you spend on coffee to go (in ¢), and $Y$ represent the amount you spend on coffee to stay (in ¢).

Then we can write the random variable $Y$ in terms of random variable $X$,

$$\begin{aligned} Y &= X + 0.085X + 50 \\ &= 1.085X + 50 \end{aligned}$$

# Example - Coffee, cont.

We know that $E(X) = 140$, $SD(X) = 30$, and $Y = 1.085X + 50$,

# Example - Coffee and a Muffin

The average price of a cup of coffee is \$1.40, with a standard deviation of 30¢. The average price of a muffin is \$2.50, with a standard deviation of 15¢. If you get a cup of coffee and a muffin every day for breakfast, what is the mean, variance, and standard deviation of the amount you spend on breakfast daily? Assume that the price of coffee and muffins are independent.

Let $X$ represent the amount you spend on coffee (in ¢), and $Y$ represent the amount you spend on muffins (in ¢).

# Simplifying RVs

Random variables do not work like normal algebraic variables:

$$X + X \neq 2X$$

$$X_1 + X_2 \neq 2X$$

If we know that $X_1$ and $X_2$ have the same distribution then,

$$\begin{aligned} E(X_1 + X_2) &= E(X_1) + E(X_2) \\ &= 2E(X_1) \end{aligned}$$

$$\begin{aligned} Var(X_1 + X_2) &= Var(X_1) + Var(X_2) \\ &= 2\ Var(X_1) \end{aligned}$$

$$\begin{aligned} E(X_1 + X_1) &= E(2X_1) \\ &= 2E(X_1) \end{aligned}$$

$$\begin{aligned} Var(X_1 + X_1) &= Var(2X_1) \\ &= 2^2\ Var(X_1) \\ &= 4\ Var(X_1) \end{aligned}$$

# Town Cars

A company has 5 Lincoln Town Cars in its fleet. Historical data show that annual fuel cost for each car is on average \$2,154 with a standard deviation of \$132. What is the mean and the standard deviation of the total annual fuel cost for this fleet?

# Pipetting

A flask is filled with exactly 100 ml of solution, you proceed take 8 ml samples via a poorly calibrated pipette. If the expected amount you pipette each time is 8 ml with a standard deviation of 0.25 ml, how much solution is left in the flask (and what is your uncertainty)?

# Properties of Binomial RVs (again)

We can also think of a Binomial random variable as the sum of independent Bernoulli random variables.

Let $X \sim \text{Binom}(n, p)$ then $X = \sum_{i=1}^{n} Y_i$ where $Y_1, \ldots, Y_n \sim \text{Bern}(p)$.