Lecture 15 - Tests of Two Proportions

Sta102 / BME102

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Colin Rundel

Difference of two proportions

Example - Melting ice cap survey

Scientists predict that global warming may have big effects on the polar regions within the next 100 years. One of the possible effects is that the northern ice cap may completely melt. Would this bother you a great deal, some, a little, or not at all if it actually happened?

(a) A great deal

(b) Some

(c) A little

(d) Not at all

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The same question was asked of 88 Duke students, of which 56 said it would bother them a great deal.

We will collapse the data such that we consider only the responses of a great deal and its compliment, not a great deal.

Collapsed Results

	US	Duke	Total
A great deal	454	56	510
Not a great deal	226	32	258
Total	680	88	768

This is an example of a contingency table (specifically a 2 x 2 contingency table).

Collapsed Results

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This is an example of a contingency table (specifically a 2 x 2 contingency table).

We are interested in comparing proportion of Duke students who say it would both them a gread deal ($p_{GD|Duke} = 56/88$) to the proportion of all Americans who say it would both them a gread deal ($p_{GD|US} = 454/680$).

Condition on what?

Knowing which of the two variables to condition on can be tricky some times.

Ask yourself - which of the two variables is most likely the dependent variable (y) and which is most likely the independent variable (x). In other words, changes in x should *cause* changes in y (not the other way around).

Once we know this then the two proportions of interest are:

dep

$$p_{y_1|x_1} \quad \text{and} \quad p_{y_1|x_2}$$

ind ind

Parameter and point estimate

- *Parameter of interest:* Difference between the proportions of *all* Duke students and *all* Americans who would be bothered a great deal by the northern ice cap melting.

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$$p_{GD|Duke} - p_{GD|US}$$

- *Point estimate:* Difference between the proportions of *sampled* Duke students and *sampled* Americans who would be bothered a great deal by the northern ice cap melting.

$$\hat{p}_{GD|Duke} - \hat{p}_{GD|US}$$

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- HT: $\textit{Test Statistic} = \frac{\textit{point estimate} - \textit{null value}}{\textit{std error}}$, find appropriate p-value using sampling distribution.
- We just need to figure out the appropriate sampling distribution and its parameters..

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We can combine that result with the approach we used for the test of two means to find the distribution of $\hat{p}_1 - \hat{p}_2$

$$\hat{p}_1 - \hat{p}_2 \sim N\big(\mu = E(\hat{p}_1 - \hat{p}_2),\ \sigma^2 = Var(\hat{p}_1 - \hat{p}_2)\big)$$

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$$\begin{aligned} E(\hat{p}_1 - \hat{p}_2) &= E(\hat{p}_1) - E(\hat{p}_2) \\ &= p_1 - p_2 \end{aligned}$$

$$\begin{aligned} Var(\hat{p}_1 - \hat{p}_2) &= Var(\hat{p}_1) + Var(\hat{p}_2) \\ &= \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n} \end{aligned}$$

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$$\hat{p}_1 - \hat{p}_2 \sim N\left(\mu = E(\hat{p}_1 - \hat{p}_2),\ \sigma^2 = Var(\hat{p}_1 - \hat{p}_2)\right)$$

$$= p_1 - p_2 \qquad = \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}$$

$$E(\hat{p}_1 - \hat{p}_2) = E(\hat{p}_1) - E(\hat{p}_2)$$
$$= p_1 - p_2$$

$$Var(\hat{p}_1 - \hat{p}_2) = Var(\hat{p}_1) + Var(\hat{p}_2)$$
$$= \frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n}$$

Note - as with the test of two means, this result requires that $\hat{p}_1$ and $\hat{p}_2$ are independent.

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2. *Independence between groups:* The sampled Duke students and the US residents are independent of each other.
3. *Success-failure:*

   At least 10 observed successes and 10 observed failures in *both* groups.

CI for difference of proportions

Construct a 95% confidence interval for the difference between the proportions of Duke students and Americans who would be bothered a great deal by the melting of the northern ice cap $(p_{GD|Duke} - p_{GD|US})$.

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A great deal	56	454
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$$\hat{p}_{GD|Duke} = 56/88 = 0.636$$
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$$SE \approx \sqrt{\frac{\hat{p}_{GD|Duke}(1 - \hat{p}_{GD|Duke})}{n_{Duke}} + \frac{\hat{p}_{GD|US}(1 - \hat{p}_{GD|US})}{n_{US}}}$$
$$= \sqrt{\frac{0.636(1 - 0.636)}{88} + \frac{0.668(1 - 0.668)}{680}} = 0.0537$$

CI for difference of proportions, cont.

$$\hat{p}_{GD|Duke} = 0.636$$
$$\hat{p}_{GD|US} = 0.668$$
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CI for difference of proportions, cont.

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\hat{p}_{GD|Duke} &= 0.636 \\
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$$

$$
\begin{aligned}
CI &= PE \pm CV \times SE \\
&= (\hat{p}_{GD|Duke} - \hat{p}_{GD|US}) \pm Z^* \times \sqrt{\frac{\hat{p}_{GD|Duke}(1 - \hat{p}_{GD|Duke})}{n_{Duke}}} \\
&= (0.636 - 0.668) \pm 1.96 \times 0.0537 \\
&= (-0.138, 0.074)
\end{aligned}
$$

CI for difference of proportions, cont.

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\end{aligned}$$

What conclusion should we draw here?

Hypotheses for testing the difference of two proportions

Just like the other hypothesis tests we have seen thus far, we formulate our null and alternative hypotheses for testing if the proportion of all Duke students who would be bothered a great deal by the melting of the northern ice cap differs from the proportion of all Americans who do as follows,

$$H_0 : p_{GD|Duke} = p_{GD|US} \quad \Rightarrow \quad p_{GD|Duke} - p_{GD|US} = 0$$

$$H_A : p_{GD|Duke} \neq p_{GD|US} \quad \Rightarrow \quad p_{GD|Duke} - p_{GD|US} \neq 0$$

Flashback to working with one proportion

When constructing a confidence interval for a population proportion, we check if the *observed* number of successes and failures are at least 10.

$$n\hat{p} \geq 10 \qquad\qquad n(1 - \hat{p}) \geq 10$$

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$$n\hat{p} \geq 10 \qquad\qquad n(1 - \hat{p}) \geq 10$$

When conducting a hypothesis test for a population proportion, we check if the *expected* number of successes and failures are at least 10.

$$np_0 \geq 10 \qquad\qquad n(1 - p_0) \geq 10$$

A slight wrinkle ...

In setting the null hypothesis for comparing two proportions we haven't fixed either $p_{GD|Duke}$ or $p_{GD|US}$ - instead we have fixed their difference.

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As such, we don't have a specific null value we can use to calculated the *expected* number of successes and failures in each group *or* the standard error. So, we know the following

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p_{GD|Duke} &= p_{GD|US} \\
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$$p_{GD|Duke} = p_{GD|US}$$
$$p_{GD|Duke} = ?$$
$$p_{GD|US} = ?$$

Does this null give us any additional useful information?

Proportions and Probabilities

Think about the sample proportions as probabilities, what does it mean if

$$P(GD|Duke) = P(GD|US)$$

Var 1 $\perp$ Var 2

(GD, NGD) (US, Duke)

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Think about the sample proportions as probabilities, what does it mean if

$$P(GD|Duke) = P(GD|US)$$

If these two probabilities are equal then global warming concern is *independent* of the Duke vs. US grouping. Which means that,

$$P(GD|Duke) = P(GD|US) = P(GD)$$

Pooling

As such, our null hypothesis is equivalent to claiming that our two categorical variables are independent. So when conducting the hypothesis test we assume the null hypothesis to be true, which means we must also assume that the two variables are independent.

Under the assumption of independence our best guess for both $p_{GD|Duke}$ and $p_{GD|US}$ will be $\hat{p}_{GD}$, which is the sample proportion of *all* respondents (from Duke or US) who answered "A great deal".

We call this value $\hat{p}_{pooled}$,

$$\hat{p}_{pooled} = \frac{\text{\# of successes in 1} + \text{\# of successes in 2}}{n_1 + n_2} = \frac{n_1\hat{p}_1 + n_2\hat{p}_2}{n_1 + n_2}$$

Pooled estimate of a proportion

Calculate the estimated pooled proportion of Duke students and Americans who would be bothered a great deal by the melting of the northern ice cap.

	Duke	US	Total
A great deal	56	454	510
Not a great deal	32	226	258
Total	88	680	788

$$\hat{p}_{pooled} = \frac{56 + 454}{88 + 680} = \frac{510}{788} = 0.664$$

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$$\hat{p}_{pooled} = \frac{56 + 454}{88 + 680} = \frac{510}{788} = 0.664$$

Which sample proportion ($\hat{p}_{GD|Duke}$ or $\hat{p}_{GD|US}$) is closer to the pooled estimate? Why?

Implications for the SE

Under the null hypothesis we are stating that $p_1 = p_2$ which does not uniquely identify either p_1 or p_2. Therefore we are using the pooled proportion ($\hat{p}$) as our best guess for p_1 and p_2 under the null hypothesis.

For a *confidence interval* we use $\hat{p}_1$ and $\hat{p}_2$ to approximate for p_1 and p_2

$$SE = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}} \approx \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$$

While for a *hypothesis test* we use $\hat{p}_{pooled}$ to approximate for p_1 and p_2

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$$= \sqrt{\hat{p}_p(1-\hat{p}_p)\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}$$

HT for comparing proportions

Do these data suggest that the proportion of all Duke students who would be bothered a great deal by the melting of the northern ice cap differs from the proportion of all Americans who do?

$$\hat{p}_{pooled} = 0.664, \quad n_1 = 88, \quad n_2 = 680$$

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$$Z = \frac{(56/88 - 454/680) - 0}{0.0535} = -0.59$$

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$$Z = \frac{(56/88 - 454/680) - 0}{0.0535} = -0.59$$

$$\text{p-value} = P(Z < -0.59 \text{ or } Z > 0.59)$$
$$= 0.277 + 0.277 = 0.555$$

HT and CI agreement

Confidence interval:

$$CI = (-0.138, 0.074)$$

Hypothesis test:

$$H_0 : p_{GD|Duke} = p_{GD|US} \qquad Z = -0.59$$

$$H_A : p_{GD|Duke} \neq p_{GD|US} \qquad \text{p-value} = 0.555$$

Do the results of the Confidence interval and hypothesis test agree? Do the necessarily have to agree?

Picking successes?

What would happen to our analysis if we had picked “Not a great deal”?

	Duke	US	Total
A great deal	56	454	510
Not a great deal	32	226	258
Total	88	680	788

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$$H_0 : p_{NGD|Duke} \neq p_{NGD|US}$$

$$\hat{p}_{pooled} = \frac{32 + 226}{88 + 680} = \frac{258}{788} = 0.336$$

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$$\hat{p}_{pooled} = \frac{32 + 226}{88 + 680} = \frac{258}{788} = 0.336$$

$$SE = \sqrt{0.336(1 - 0.336)\left(\frac{1}{88} + \frac{1}{680}\right)} = 0.0535$$

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$$SE = \sqrt{0.336(1 - 0.336)\left(\frac{1}{88} + \frac{1}{680}\right)} = 0.0535$$

$$Z = \frac{(32/88 - 226/680) - 0}{0.0535} = 0.585$$

$$\text{p-value} = P(Z < -0.59 \text{ or } Z > 0.59) = 0.277 + 0.277 = 0.555$$

Swapping dependent and independent variables?

What would happen to our analysis if we had swapped our independent and dependent variable?

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$$H_0 : p_{Duke|GD} = p_{Duke|NGD}$$

$$H_0 : p_{Duke|GD} \neq p_{Duke|NGD}$$

$$\hat{p}_{pooled} = \frac{56 + 32}{510 + 258} = \frac{88}{788} = 0.115$$

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$$H_0 : p_{Duke|GD} \neq p_{Duke|NGD}$$

$$\hat{p}_{pooled} = \frac{56 + 32}{510 + 258} = \frac{88}{788} = 0.115$$

$$SE = \sqrt{0.115(1 - 0.115)\left(\frac{1}{510} + \frac{1}{258}\right)} = 0.0241$$

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$$H_0 : p_{Duke|GD} \neq p_{Duke|NGD}$$

$$\hat{p}_{pooled} = \frac{56 + 32}{510 + 258} = \frac{88}{788} = 0.115$$

$$SE = \sqrt{0.115(1 - 0.115)\left(\frac{1}{510} + \frac{1}{258}\right)} = 0.0241$$

$$Z = \frac{(56/510 - 32/258) - 0}{0.0241} = 0.59$$

$$\begin{aligned} \text{p-value} &= P(Z < -0.59 \text{ or } Z > 0.59) \\ &= 0.2775 + 0.2775 = 0.555 \end{aligned}$$

Power

What it the power of our hypothesis test to detect a difference of +0.1?

Step 0: $H_0 : p_1 = p_2$ $H_A : p_1 \neq p_2$ $\alpha = 0.05$ $\delta = +0.1$ $SE = 0.0535$ power $= ?$

Step1: Find ze such that we reject H_0

$$P(Z > 2 \text{ or } Z < -2) < 0.05$$

$$Z > 1.96 \quad \text{or} \quad Z < -1.96$$

$$\frac{\hat{p}_1 - \hat{p}_2 - 0}{0.0535} > 1.96 \qquad \frac{\hat{p}_1 - \hat{p}_2}{0.0535} < -1.96$$

$$\hat{p}_1 - \hat{p}_2 > 0.105 \qquad \hat{p}_1 - \hat{p}_2 < -0.105$$

Step 2: Assume H_A is true

H_A $p_1 - p_2 = \delta = 0.1 \Rightarrow$ not indep

$$SE \approx \sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}} = 0.0537$$

$$P\left(\hat{p}_1 - \hat{p}_2 > 0.105 \text{ or } \hat{p}_1 - \hat{p}_2 < -0.105 \,\middle|\, p_1 - p_2 = 0.1\right)$$

$$= P\left(Z > \frac{0.105 - 0.1}{0.0537} \text{ or } Z < \frac{-0.105 - 0.1}{0.0537}\right)$$

$$= P(Z > 0.094) + P(Z < -3.88)$$

$$= 0.462 + 0 = 0.462 \Rightarrow \text{Power}$$

Power

What it the power of our hypothesis test to detect a difference of 0.1?

Step 0: $H_0 : p_1 = p_2 \quad H_A : p_1 \neq p_2 \quad \alpha = 0.05 \quad \delta = 0.1 \quad SE = 0.0535 \quad \text{power} =?$

Step 1: Find $\hat{p}_1 - \hat{p}_2$ such that we reject H_0.

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Step 1: Find $\hat{p}_1 - \hat{p}_2$ such that we reject H_0.

$$P(Z < -z \text{ or } Z > z) < 0.05 \Rightarrow z > 1.96$$

Power

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Step 1: Find $\hat{p}_1 - \hat{p}_2$ such that we reject H_0.

$$P(Z < -z \text{ or } Z > z) < 0.05 \Rightarrow z > 1.96$$

$$\frac{(\hat{p}_1 - \hat{p}_2) - 0}{0.0535} < -1.96 \quad \text{or} \quad \frac{(\hat{p}_1 - \hat{p}_2) - 0}{0.0535} > 1.96$$

Power

What it the power of our hypothesis test to detect a difference of 0.1?

Step 0: $H_0 : p_1 = p_2 \quad H_A : p_1 \neq p_2 \quad \alpha = 0.05 \quad \delta = 0.1 \quad SE = 0.0535 \quad \text{power} = ?$

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Step 2: Assume $p_1 - p_2 = 0 + \delta = 0.1$ - we no longer assume independence, must use $SE = 0.0537$ from the CI instead.

$$\begin{aligned} &P(\hat{p}_1 - \hat{p}_2 < -0.105 \quad \text{or} \quad \hat{p}_1 - \hat{p}_2 > 0.105) \\ &= P\left(Z < \frac{-0.105 - 0.1}{0.0527}\right) + P\left(Z > \frac{0.105 - 0.1}{0.0527}\right) \\ &= P(Z < -3.88) + P(Z > 0.094) \end{aligned}$$

Example - Planned Parenthood

Planned Parenthood

A Pew Research poll conducted between September 22-27, 2015 asked 805 randomly sampled Americans (who self identify as a Democrat or Republican) and ask about their party affiliation and whether they think any budget agreement must eliminate or maintain funding for Planned Parenthood. The distribution of their responses is shown below.

	Eliminate	Maintain	Total
Democrat	45	378	423
Republican	277	105	382
Total	322	483	805

Pew Research Center. *Majority Says Any Budget Deal Must Include Planned Parenthood Funding.* Sep 28, 2015. http://www.people-press.org/2015/09/28/majority-

Analysis - HT

Is there evidence that a greater percentage of Democrats support maintaining funding for Planned Parenthood than Republicans?

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$$\hat{p}_{pooled} = \frac{378 + 105}{423 + 382} = 0.6$$
$$SE = \sqrt{0.6(1 - 0.6)\left(\frac{1}{423} + \frac{1}{382}\right)} = 0.0346$$

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$$Z = \frac{(378/423 - 105/382) - 0}{0.0346} = 17.88$$

$$\text{p-value} = P(Z > 17.88) \approx 0$$

Analysis - CI

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$$\begin{aligned} CI &= (\hat{p}_{m|D} - \hat{p}_{m|R}) \pm Z^* \; SE \\ &= (378/423 - 105/382) \pm 1.96 \times 0.0273 \\ &= (0.565,\; 0.672) \end{aligned}$$

Power

What it the power of our hypothesis test to detect a difference of 0.62?

Step 0: $H_0 : p_1 = p_2 \quad H_A : p_1 > p_2 \quad \alpha = 0.05 \quad \delta = 0.62 \quad SE = 0.0346 \quad \text{power} = ?$

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Step 2: Assume $p_1 - p_2 = 0 + \delta = 0.62$ - we no longer assume independence, must use $SE = 0.0273$ from the CI instead.

$$\begin{aligned} &P\left(\hat{p}_1 - \hat{p}_2 > 0.0569\right) \\ &= P\left(Z > \frac{0.0569 - 0.62}{0.0273}\right) \\ &= P\left(Z > -20\right) \end{aligned}$$

Recap

Recap - inference for one proportion

- Population parameter: p, point estimate: $\hat{p}$

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- Standard error: $SE = \sqrt{\frac{p(1-p)}{n}}$
  - for CI: use $\hat{p}$
  - for HT: use p_0
  - for Power:
    - Step 1 - use p_0
    - Step 2 - use $p_A = p_0 + \delta$

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    - *observed* for CI
    - *expected* for HT
- $SE = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$
  - for CI: use $\hat{p}_1$ and $\hat{p}_2$
  - for HT:
    - when $H_0 : p_1 = p_2$: use $\hat{p}_{pool} = \frac{\#suc_1 + \#suc_2}{n_1 + n_2}$
    - when $H_0 : p_1 - p_2 = (\textit{some value other than 0})$: use $\hat{p}_1$ and $\hat{p}_2$
      - this is pretty rare
  - for Power:
    - Step 1 - use $\hat{p}_{pool}$
    - Step 2 - use $\hat{p}_1$ and $\hat{p}_2$

Reference - standard error calculations

	one sample	two samples
mean	$SE = \frac{\sigma}{\sqrt{n}}$	$SE = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$
proportion	$SE = \sqrt{\frac{p(1-p)}{n}}$	$SE = \sqrt{\frac{p_1(1-p_1)}{n_1} + \frac{p_2(1-p_2)}{n_2}}$

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- When working with means, it's very rare that σ is known, so we usually use s as an approximation.
- When working with proportions, we will not know p therefore
  - if doing a hypothesis test, p comes from the null hypothesis
  - if constructing a confidence interval, use $\hat{p}$ instead