Lecture 4 - Conditional Probability

Sta102 / BME102

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Probability Review

Basic Probability Review

Defn. of Probability	$0 \leq P(A) \leq 1$
Complement Rule	$P(A^c) = 1 - P(A)$
General Include-Exclusion Rule	$P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Multiplication Rule	$P(A \cap B) = P(A\|B) \times P(B) = P(B\|A) \times P(A)$
Bayes' Rule	$P(B\|A) = \frac{P(A\|B)P(B)}{P(A)}$
Law of Total Probability	$P(A) = P(A\|B)P(B) + P(A\|B^c)P(B^c)$

Other Important Terms

Joint probability of A and B	$P(A \cap B)$
Marginal probability of A	$P(A)$
Conditional probability of A given B	$P(A\|B)$

Disjoint and Independent

Your intuition can easily be wrong on this type of problem, always check using the definitions:

A and B are *independent* iff:

$$P(A \cap B) = P(A) \times P(B)$$

$$\text{or } P(A|B) = P(A)$$

$$\text{or } P(B|A) = P(B)$$

A and B are *disjoint* (mutually exclusive) iff:

$$P(E \cap F) = 0$$

Sampling and Probability

Sampling

Imagine an urn filled with white and black marbles

. . . or a deck of cards

. . . or a bingo cage

. . . or a hat full of raffle tickets

Two common options + one extra for completeness:

- Sampling without replacement
- Sampling with replacement
- Pólya's urn

Quick Examples (1)

What is the probability of being dealt two aces?

$n = 52 \quad \# A = 4$

$P(2 \text{ aces}) = P(D_1 = A, D_2 = A)$

$= P(D_1 = A) P(D_2 = A \mid D_1 = 1)$

$= \frac{4}{52} \frac{3}{51} = 0.004$

Quick Examples (2)

What if you replace the first card before reshuffling and drawing the second card?

$$P(d_1 = A, d_2 = A)$$

$$= P(d_1 = A)\, P(d_2 = A \mid \cancel{d_1 = A})$$

$$= 4/52 \times 4/52$$

Quick Examples (3)

What if you replace the first card and added an additional copy of that card before reshuffling and drawing the second card?

$$P(C_1 = A, C_2 = A)$$

$$= P(C_1 = A)\, P(C_2 = A \mid C_1 = A)$$

$$= 4/52 \quad 5/53$$

Harder Example

What is the probability of being dealt a royal flush in poker?

(Ace, king, queen, jack, 10 of the same suit in any order)

$$P(rf) = 4 \times P(\text{royal flush of spades})$$

$$= 4 \times \frac{5}{52} \frac{4}{51} \frac{3}{50} \frac{2}{49} \frac{1}{48}$$

$$= 0.000000384$$

$$= \frac{20}{52} \frac{4}{51} \frac{3}{50} \cdot \frac{2}{49} \frac{1}{48}$$

$$P(rf) = 4 \cdot P(A, K, Q, J, 10)$$

$$= 4 \times P(d_1 = \{A, K, Q, J, 10\} \text{ of } \diamond)$$

$$P(d_2 = \{A, K, Q, J, 10 \text{ but not } d_1\} \mid d_1)$$

$$\cdots$$

$not(d_1 \ or \ d_2)$

$not \ d_1 \ and \ not \ d_2$

And now a brief magic trick ...

If you have ever shuffled a deck of cards you have done something no one else has ever done before or will ever do again ...

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$$52 \cdot 51 \cdot 50 \cdots 2 \cdot 1 = 52!$$

To put that in context:

- Cells in the human body (10^{14})
- Seconds since the big bag (10^{18})
- Grains of sand on all beaches on earch (7.5×10^{18})
- Stars in the universe (10^{23})
- Atoms in the observable universe (10^{80})
- A Googol (10^{100})

Birthday Problem

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Let A_i be the event that student i does not match any of the preceding students then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2|A_1)\cdots P(A_n|A_1,\ldots,A_{n-1})$$

Birthday Problem, cont.

$$P(\text{no match}) = \underbrace{\frac{365}{365}}_{\text{Person 1}} \underbrace{\frac{364}{365}}_{\text{Per. 2}} \underbrace{\frac{363}{365}}_{\text{Per 3}} \cdots \underbrace{\frac{300}{365}}_{\text{Per 66}}$$

$$P(\text{no match per } n \mid p_1 \ldots p_{n-1}) = \frac{365-(n-1)}{365}$$

$P(\text{no match in } n \text{ people})$

$$= \prod_{i=1}^{n} \frac{365-(n-1)}{365}$$

$$= \frac{365!/(365-n)!}{365^n}$$

$365 \cdot 364 \cdot 363 \cdots$

$$\frac{365!}{(365-n)!}$$

$$P(\text{no match})$$

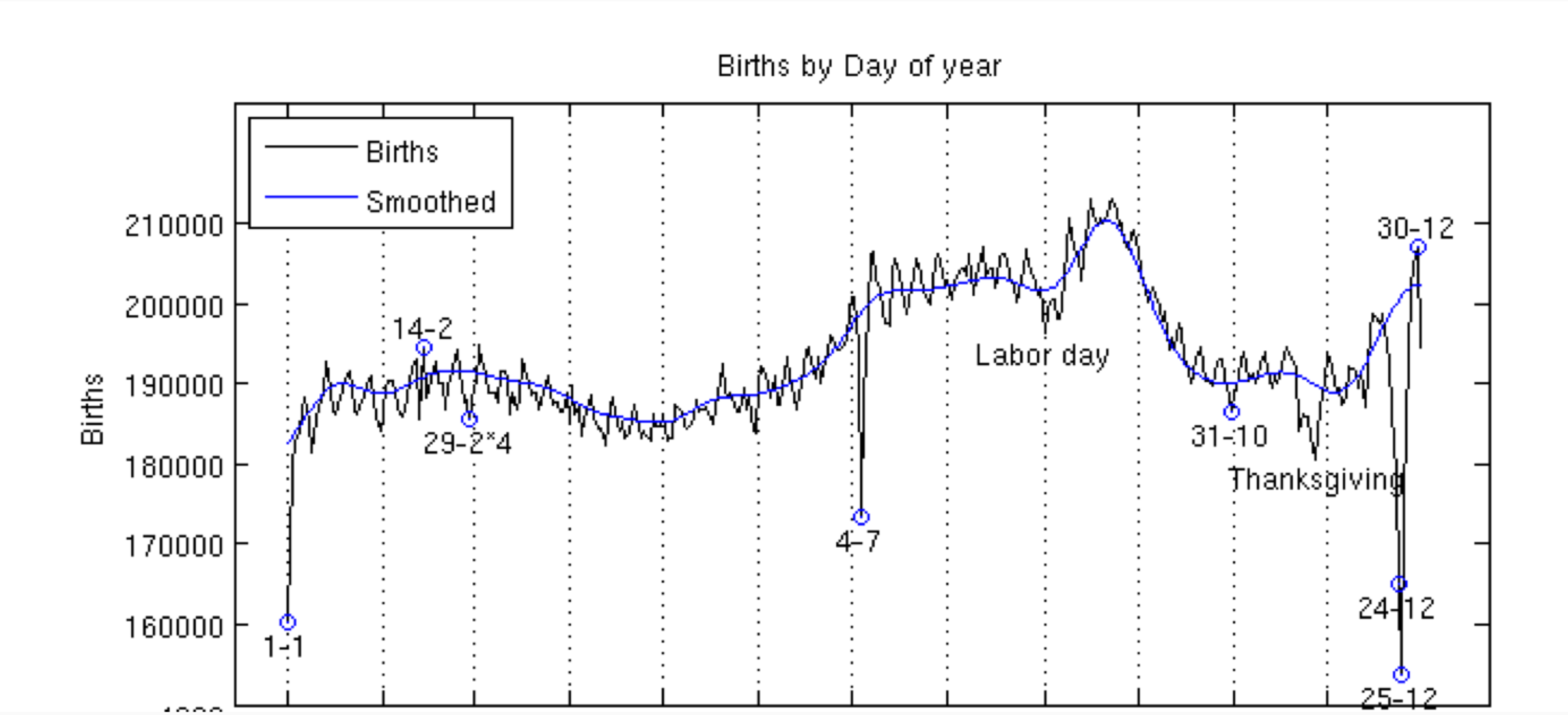
$$= P(P_1) + P(P_2 \mid P_1)$$

$$\times P(P_3 \mid P_1, P_2)$$

$$\times P(P_4 \mid P_1, P_2, P_3)$$

$$\times \ldots$$

Birthday Problem, cont.



A Bayesian Example

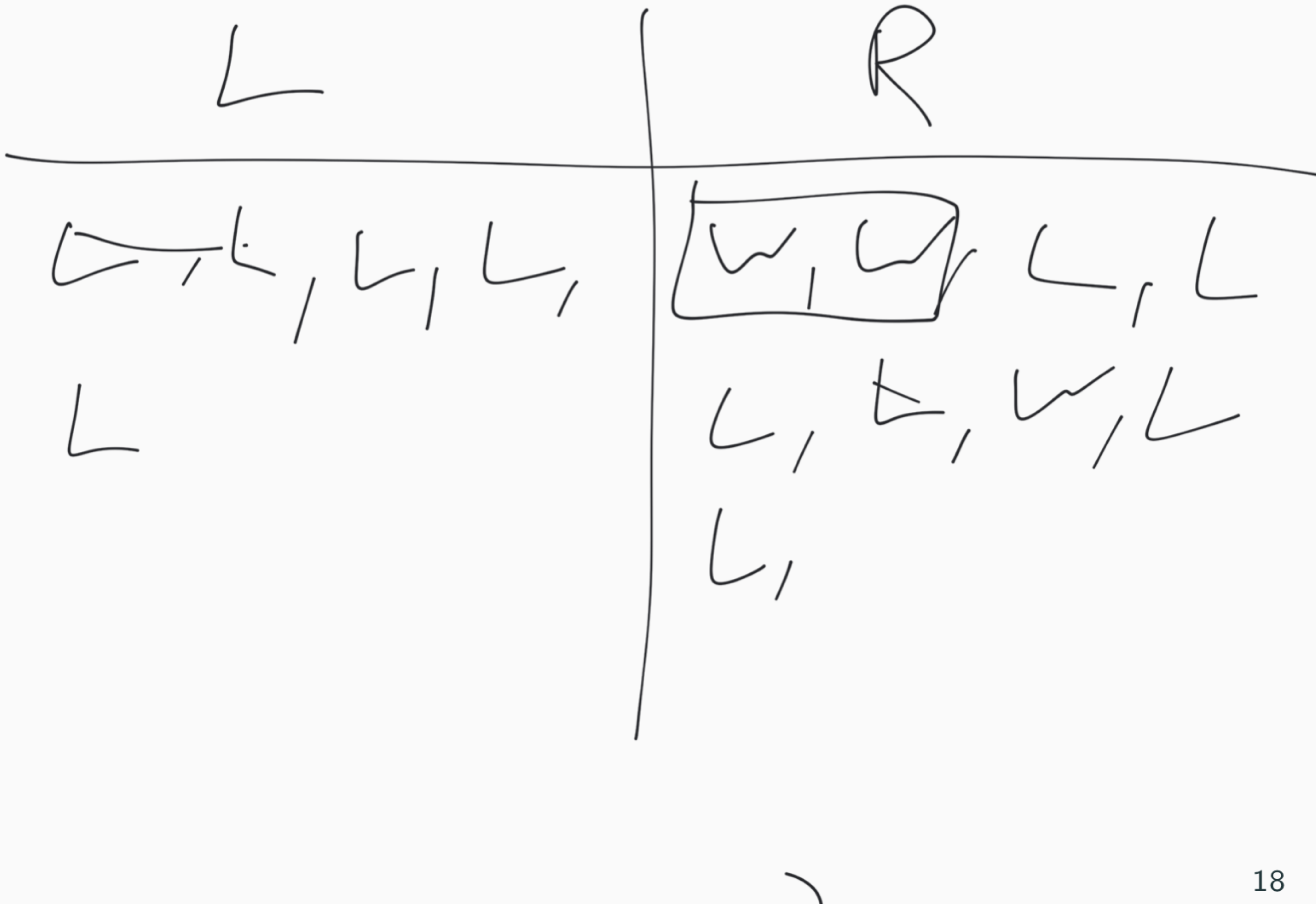
The two armed bandit ...

Imagine you walk into a casino and there are two slot machines in front of you - the pit boss walks up to you and tells you:

“I like your face, so I’m gonna help you out. One of these two machines is good and pays out 50% of the time, the other one is bad it only pays out 20% of the time.”

What should you do?

Activity Results



Probility Calculations

Looking @ right

$P(F) = 1/2 \quad P(B) = 1/2$

$P(W|F) = 1/2 \quad P(W|B) = 1/5$

$$P(F|W) = \frac{P(W|F)\,P(F)}{P(W|F)\,P(F) + P(W|B)\,P(B)}$$

$$= \frac{1/2\,(1/2)}{1/2\,(1/2) + 1/5\,(1/2)} = 5/7$$

Probility Calculations (cont.)

Update belif $\Rightarrow P(F) = 5/7$

$P(B) = 2/7$

$$P(F \mid W_1, W_2) = \frac{P(W|F)P(F)}{P(W|F)P(F) + P(W|B)P(B)}$$

$$= \frac{1}{2}(5/7) \Big/ \left(1/2 \cdot (5/7) + 1/5(2/7)\right) =$$

0.862

Update beliefs

$P(E) = 0.862$

$P(B) = 0.138$

$$P(E|L) = \frac{P(L|E)P(E)}{P(L|E)P(E) + P(L|B)P(B)}$$

Probability Calculations - the long way

If we had played the same machine twice and one both times then the probability it is the good machine is:

$$
\begin{aligned}
P(G|W_1, W_2) &= \frac{P(W_1, W_2|G)\ P(G)}{P(W_1, W_2)} \\
&= \frac{P(W_1|G)\ P(W_2|G)\ P(G)}{P(W_1)P(W_2|W_1)} \\
&= \frac{P(W_2|G)}{P(W_2|W_1)} \left[\frac{P(W_1|G)\ P(G)}{P(W_1)} \right] \\
&= \frac{P(W_2|G)}{P(W_2|W_1)} \Big[P(G|W_1) \Big] \\
&= \frac{P(W_2|G)}{P(W_2|G, W_1)P(G|W_1) + P(W_2|B, W_1)P(B|W_1)} \Big[P(G|W_1) \Big] \\
&= \frac{P(W_2|G)}{P(W_2|G)P(G|W_1) + P(W_2|B)P(B|W_1)} \Big[P(G|W_1) \Big] \\
&= \frac{1/2}{1/2 \times 5/7 + 1/5 \times 2/7}\ \frac{5}{7} = 25/29 = 0.862
\end{aligned}
$$

θ – Quantity of interest

X – data

$$P(\theta|X) = \frac{P(X|\theta)}{P(X)} P(\theta)$$

$P(X|\theta)$ → Likelihood

$P(\theta|X)$ → Posterior

$P(\theta)$ → Prior

Why do we care?

The two-armed (multi-armed) bandit is a very useful model when it comes to clinical trials.

We are trying one or more treatments against a control and we want to know the efficacy of those treatments. This is much more complex in practice because not only do we not know which is better ($P(G)$ in our slot example) we also don't know how much better they are (also need to estimate $P(W|G)$).

Complex optimization problem where we must allocation a limited number of subjects to properly balance:

- Exploration - estimate the payoff of each treatment
- Exploitation - get the best outcome for the most patients

Back to House and Lupus

Last time we worked through a problem on the probability of a patient having lupus given they test positive. We were given

- $P(L) = 0.02$
- $P(+|L) = 0.99$
- $P(-|L^c) = 0.74$

From which we calculated that

$$P(L|+) = \frac{P(L \cap +)}{P(+)} = \frac{P(+|L)P(L)}{P(+|L)P(L) + P(+|L^c)P(L^c)} = \frac{0.02 \times 0.99}{0.02 \times 0.99 + 0.98 \times 0.26} = 0.072$$

If the patient gets a second test, how should our belief in the probability of having lupus, $P(L)$, change?

Another Example

Let's Make a Deal. . .

Monty Hall Problem

You are offered a choice of three doors, there is a car behind one of the doors and there are goats behind the other two.

Monty Hall, Let's Make a Deal's original host, lets you choose one of the three doors.

Monty then opens one of the other two doors to reveal one of the goats.

You are then allowed to stay with your original choice or switch to the other door.

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Which option should you choose?

(a) stay (b) switch (c) it does not matter

A Little History

First known formulation comes from a 1975 letter by Steve Selvin to the American Statistician.

Popularized in 1990 by Marilyn vos Savant in her "Ask Marilyn" column in Parade magazine.

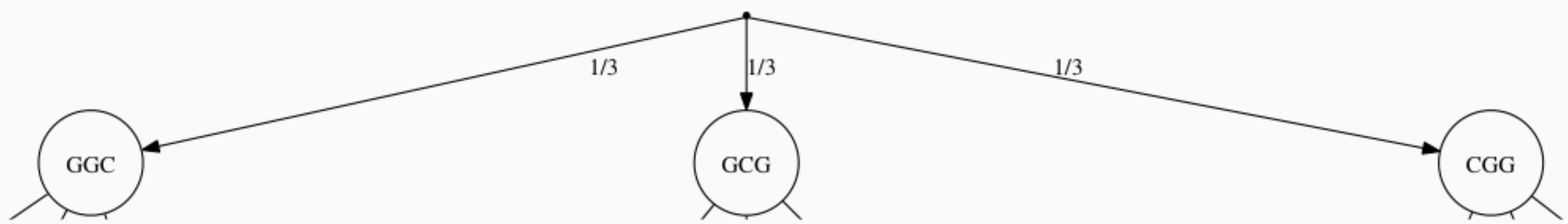
- vos Savant's solution claimed that the contestant should always switch
- About 10,000 (1,000 from Ph.D.s) letters contesting the solution
- vos Savant was right, easy to show with simulation

Moral of the story: trust the math not your intuition

A slightly more entertaining variant of Monty Hall ...

https://www.youtube.com/watch?v=tvODuUMLLgM&t=3m24s

Monty Hall - The hard way



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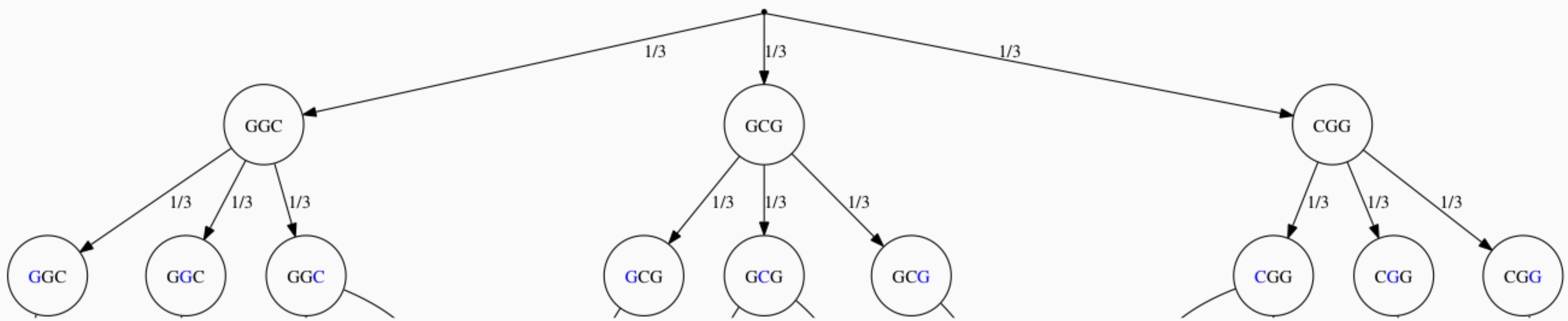

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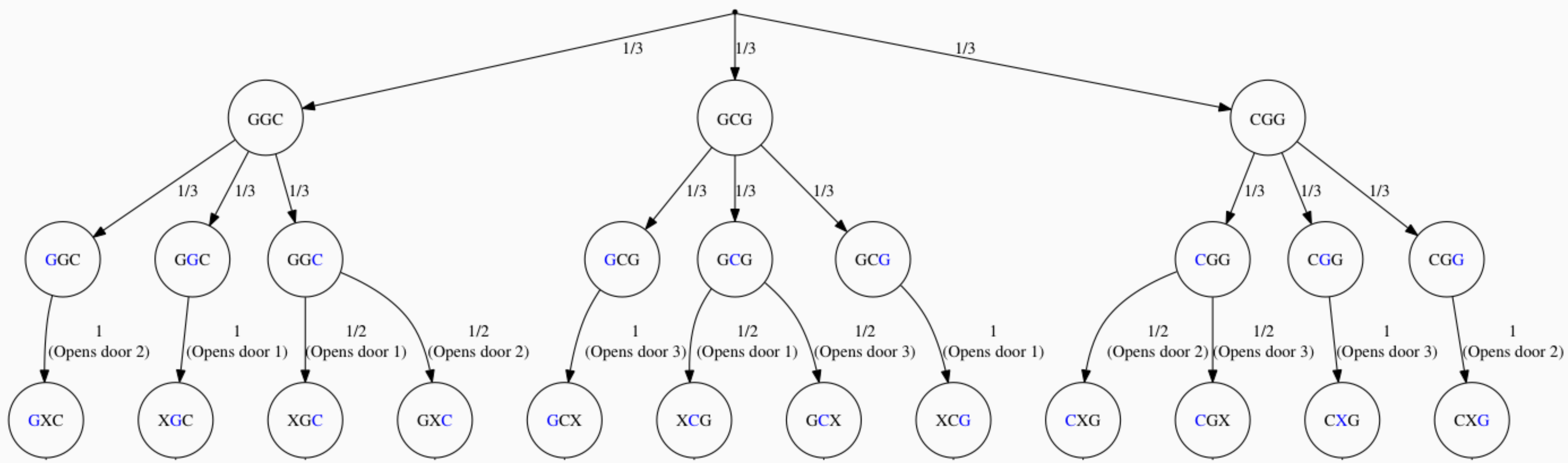
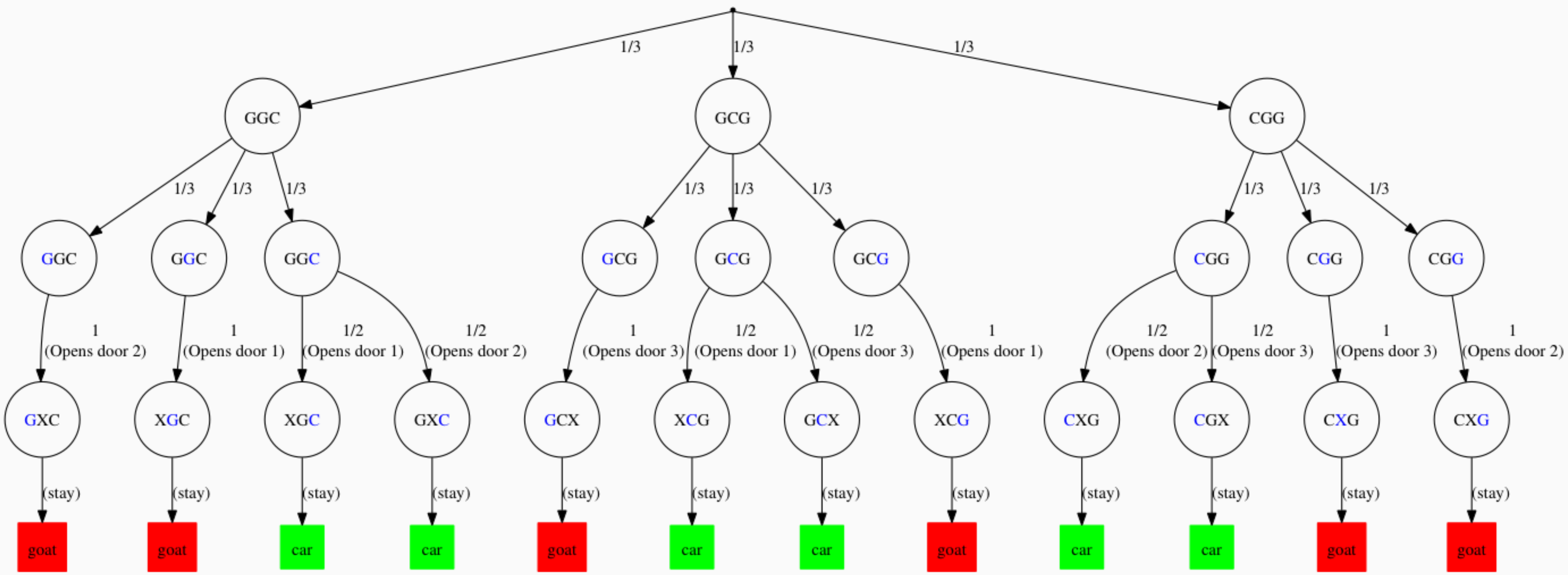
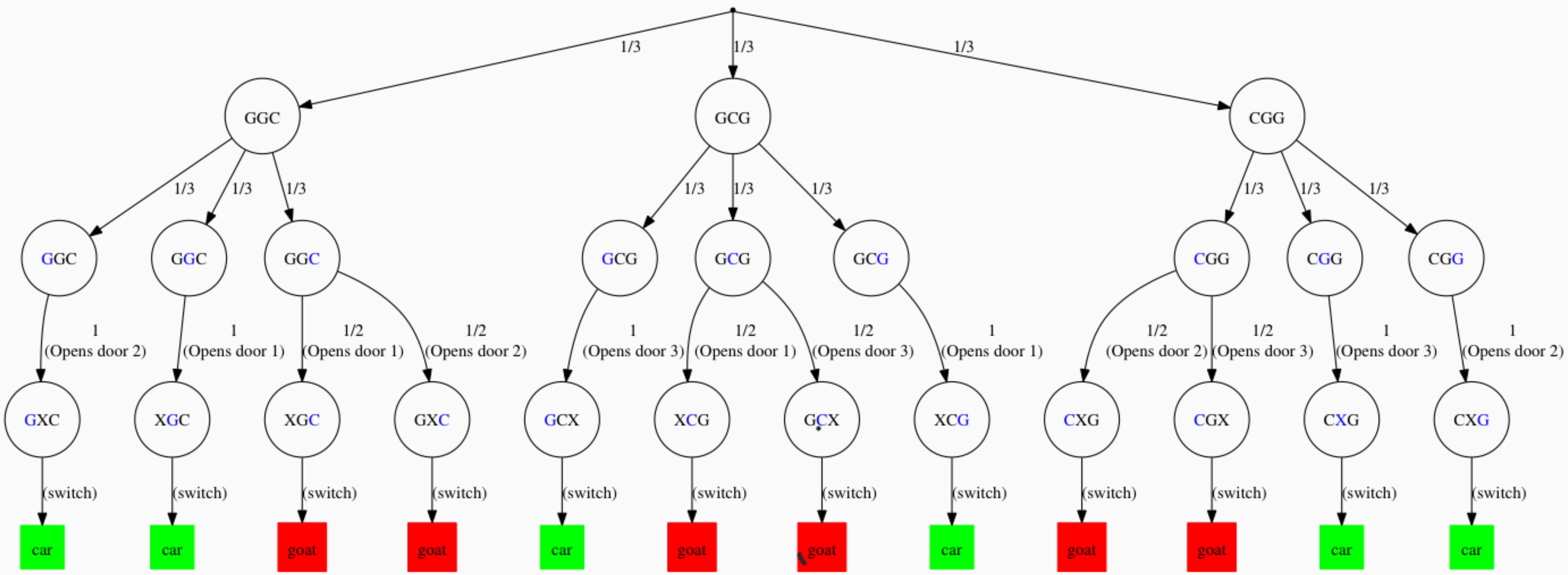


Monty Hall - The hard way - Stay



$$P(Car|Stay) = \frac{1}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{18} + \frac{1}{18} = 6/18 = 1/3$$

Monty Hall - The hard way - Switch



$$P(Car|Switch) = \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} + \frac{1}{9} = 6/9 = 2/3$$