Lecture 3 - Probability and Conditional Probability

Sta102 / BME102

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Probability

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- Belief: If you are indifferent between winning \$1 if E occurs or winning \$1 if you draw a blue chip from a box with $1,000,000 \times p$ blue chips, rest red,

$$P(E) = p$$

Terminology

Outcome space (Ω) - set of all possible outcomes (ω).

Examples:	3 coin tosses	{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
	One die roll	{1,2,3,4,5,6}
	Sum of two rolls	{2,3,. . . ,11,12}
	Time waiting for bus	$[0, \infty)$

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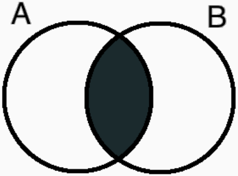
Event (E) - subset of Ω ($E \subseteq \Omega$) that might or might not happen

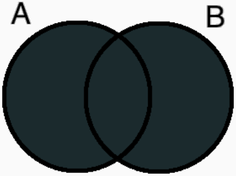
Examples:	2 heads	{HHT, HTH, THH}
	Roll an even number	{2,4,6}
	Wait $<$ 2 minutes	$[0, 120)$

Set Operations

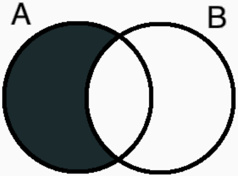
Intersection - A and B



Union - A or B



Complement - not B



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- All probabilities are between 0 and 1
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  $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
- Complement Rule:
  $P(\text{not } A) = P(A^c) = 1 - P(A)$

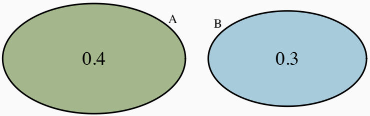
Disjoint vs. non-disjoint events

disjoint events:

P(A or B)

= P(A) + P(B) - P(A and B)

= 0.4 + 0.3 - 0 = 0.7



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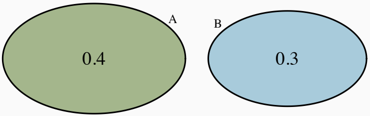
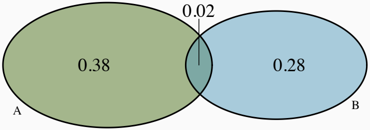


non-disjoint events:

P(A or B)

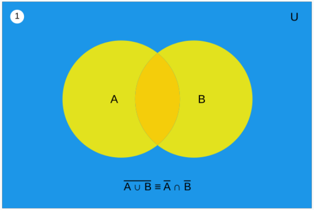
= P(A) + P(B) - P(A and B)

= 0.4 + 0.3 - 0.02 = 0.68



DeMorgan's Rules

$$P\Big(\text{not } (A \text{ or } B)\Big) = P\Big((\text{not } A) \text{ and } (\text{not } B)\Big)$$



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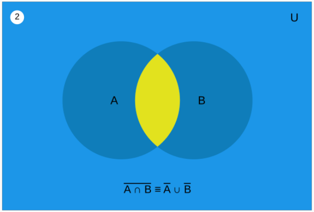
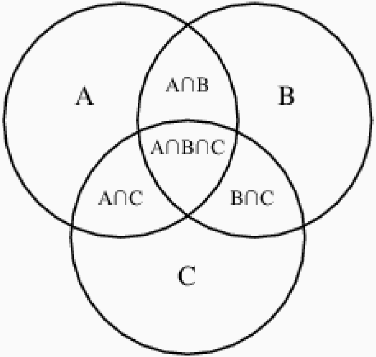


Working with three events

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \text{ and } B) - P(A \text{ and } C) - P(B \text{ and } C) + P(A \text{ and } B \text{ and } C)$$



Conditional Probability

Conditional Probability

Jackson et al. (2013) wanted to know whether it is better to give the diphtheria, tetanus and pertussis (DTaP) vaccine in either the thigh or the arm, so they collected data on severe reactions to this vaccine in children aged 3 to 6 years old. The data are summarized in the contingency table below. Is the probability of a severe reaction higher for vaccines in the thigh or the arm?

	No severe reactions	Severe reaction	Total
Thigh	4,758	30	4,788
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Total	13,598	106	13,704

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$P(\text{severe reaction} \mid \text{arm}) = 76/8,916 \approx 0.0085$

Conditional Probability, cont.

We can rewrite the counting definition of conditional probability as

$$\begin{aligned} P(A \mid B) &= \frac{\#(A \text{ and } B)}{\#(B)} \\ &= \frac{P(A \text{ and } B)}{P(B)} \end{aligned}$$

which is the general definition of conditional probability.

Note that $P(A|B)$ is undefined if $P(B) = 0$.

Multiplication rule

Very often we may know the probability of events and their conditional probabilities but not probabilities of the events together, in which case we can use

$$P(A \text{ and } B) = P(A \mid B) \times P(B)$$

Example - Hiking

Whether or not I go hiking depends on the weather, if it is sunny there is a 60% chance I will go for a hike, while there is only a 10% chance if it is raining and a 30% chance if it is snowing.

The weather forecast for tomorrow calls for 50% chance of sunshine, 40% chance of rain, and a 10% chance of snow.

What is the probability I go for a hike tomorrow?

Independence

We defined events A and B to be independent when

$$P(A \text{ and } B) = P(A) \times P(B)$$

which also implies that

$$P(A \mid B) = P(A)$$
$$P(B \mid A) = P(B)$$

Disjoint vs. independent events

- *Disjoint (mutually exclusive) events* cannot happen at the same time
  - A voter cannot register as a Democrat and a Republican at the same time
  - But they might be a Republican and a Moderate at the same time – *non-disjoint events*
  - For disjoint A and B: $P(A \text{ and } B) = 0$

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  - For disjoint A and B: $P(A \text{ and } B) = 0$
- If A and B are *independent events*, having information on A does not tell us anything about B (and vice versa)
  - If A and B are independent:
    - $P(A \mid B) = P(A)$
    - $P(A \text{ and } B) = P(A) \times P(B)$

Example - Eye and hair color

Table 3.3.1 Hair color and eye color

		Hair color			
		Brown	Black	Red	Total
Eye color	Brown	400	300	20	720
	Blue	800	200	50	1,050
	Total	1,200	500	70	1,770

1. Are brown and black hair disjoint?
2. Are brown and black hair independent?
3. Are brown eyes and red hair disjoint?
4. Are brown eyes and red hair independent?

Bayes' Rule

Expands on the definition of conditional probability to give a relationship between $P(B|A)$ and $P(A|B)$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

In the case where $P(A)$ is not known we can extend this using the law of total probability

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

Example - House

If you've ever watched the TV show *House*, you know that Dr. House regularly states, "It's never lupus."

Lupus is a medical phenomenon where antibodies that are supposed to attack foreign cells to prevent infections instead see plasma proteins as foreign bodies, leading to a high risk of blood clotting. It is believed that 2% of the population suffer from this disease.

The test for lupus is very accurate if the person actually has lupus, however is very inaccurate if the person does not. More specifically, the test is 98% accurate if a person actually has the disease. The test is 74% accurate if a person does not have the disease.

Is Dr. House correct even if someone tests positive for Lupus?