Lecture 11: Conditional Distributions

Sta 111

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Conditional Distributions

Let X and Y be random variables then
Conditional probability:

$$P(X = x|Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

$$f(x|y) = \frac{f(x, y)}{f_Y(y)}$$

$$f(y|x) = \frac{f(x, y)}{f_X(x)}$$

Product rule:

$$P(X = x, Y = y) = P(X = x|Y = y)P(Y = y)$$

$$f(x, y) = f(x|y)f_Y(y)$$

Example - Constrained Sum of Poissons

Let X and Y be Poisson random variables with rates λ_1 and λ_2, what is the conditional distribution of X given $X + Y = n$. Hint - $X + Y$ will follow a Poisson distribution with rate $\lambda_1 + \lambda_2$.

Example - Manufacturing

A manufacturing process consists of two stages. The first stage takes Y minutes, and the whole process takes X minutes (which includes the first Y minutes). Suppose that X and Y have a joint continuous distribution with joint pdf

$$f(x, y) = e^{-x}, \quad \text{for } 0 < y < x < \infty$$

If we observe that Y takes 4 minutes, what is the probability that X takes longer than 9 minutes?

Example - Defective Parts

Suppose that a certain machine produces defective and nondefective parts, but we do not know what proportion of defectives we would find among all parts that could be produced by the machine. The distribution of X, assuming that we know $P = p$, is the binomial distribution with parameters n and p. Given no other information we might believe that P has a continuous distribution with pdf such as $f_P(p) = 1$ for $p \in (0, 1)$.

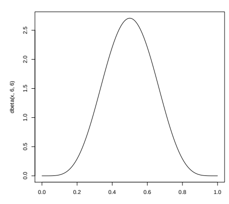
What is the joint probability of $f(x, p)$? What is the marginal distribution of X?

Example - Defective Parts, cont.

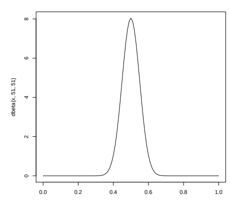
Based on the preceding results, what is the conditional distribution of P given $X = 5$ and $N = 10$?

Example - Defective Parts, cont.

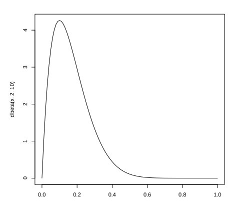
$P|X = 5, N = 10 \sim \text{Beta}(6, 6)$



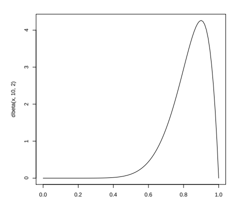
$P|X = 50, N = 100 \sim \text{Beta}(51, 51)$



$P|X = 1, N = 10 \sim \text{Beta}(2, 10)$



$P|X = 9, N = 10 \sim \text{Beta}(10, 2)$



Bayes' Theorem and the Law of Total Probability

Let X and Y be random variables then
Bayes' Theorem:

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f(y|x) f_X(x)}{f_Y(y)}$$

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{f(x|y) f_Y(y)}{f_X(x)}$$

Law of Total Probability:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_{-\infty}^{\infty} f(x|y) f_Y(y)\, dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\, dx = \int_{-\infty}^{\infty} f(y|x) f_X(x)\, dx$$

Example - Conditional Uniforms

Let $X \sim \text{Unif}(0,1)$ and $Y|X \sim \text{Unif}(x,1)$, find $f(x,y)$, $f_Y(y)$, and $f(x|y)$.

Independence and Conditional Distributions

If X and Y are independent random variables then

$$f(x|y) = f_X(x)$$
$$f(y|x) = f_Y(y)$$

Justification for this is straight forward:

$$f(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{f_X(x) f_Y(y)}{f_Y(y)} = f_X(x)$$

Conditional Expectation

If X and Y are independent random variables then we define the conditional expectation as follows

$$E(X|Y=y) = \sum_{\text{all } x} x\, f(x|y)\; dx$$

$$E(X|Y=y) = \int_{-\infty}^{\infty} x\, f(x|y)\; dx$$

Example - Family Cars (Example 4.7.2 deGroot)

Let X be the number of members in a randomly selected household and Y be the number of cars owned by that household.

Table 4.1 Reported numbers of household members and automobiles in Example 4.7.1

Number of automobiles	Number of members 1	2	3	4	5	6	7	8
0	10	7	3	2	2	1	0	0
1	12	21	25	30	25	15	5	1
2	1	5	10	15	20	11	5	3
3	0	2	3	5	5	3	2	1

Table 4.2 Joint p.f. $f(x, y)$ of X and Y in Example 4.7.2 together with marginal p.f.'s $f_1(x)$ and $f_2(y)$

y	x = 1	2	3	4	5	6	7	8	$f_2(y)$
0	0.040	0.028	0.012	0.008	0.008	0.004	0	0	0.100
1	0.048	0.084	0.100	0.120	0.100	0.060	0.020	0.004	0.536
2	0.004	0.020	0.040	0.060	0.080	0.044	0.020	0.012	0.280
3	0	0.008	0.012	0.020	0.020	0.012	0.008	0.004	0.084
$f_1(x)$	0.092	0.140	0.164	0.208	0.208	0.120	0.048	0.020	

We can then calculate the expected number of cars in a household with 4 members as follows

$$E(Y|X=4) = \sum_{y=0}^{4} y \cdot P(Y=y|X=4) = \sum_{y=0}^{4} y \cdot P(X=4, Y=y)/P(X=4)$$
$$= 0 \times 0.0385 + 1 \times 0.5769 + 2 \times 0.2885 + 3 \times 0.0962 = 1.442$$

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2	1	5	10	15	20	11	5	3
3	0	2	3	5	5	3	2	1

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$f_1(x)$	0.092	0.140	0.164	0.208	0.208	0.120	0.048	0.020	

Similarly, we can calulcate $E(Y|X)$ for the other seven values of x

x	1	2	3	4	5	6	7	8
$E(Y\|X=x)$	0.609	1.057	1.317	1.442	1.538	1.533	1.75	2

Conditional Expectation as a Random Variable

Based on the previous example we can see that the value of $E(Y|X)$ changes depending on the value of x.

As such we can think of the conditional expectation as being a function of the random variable X, thereby making $E(Y|X)$ itself a random variable, which can be manipulated like any other random variable.

Law of Total Probability for Expectations

$$\begin{aligned}
E\big[E(Y|X)\big] &= \int_{-\infty}^{\infty} E(Y|X=x)\, f_X(x)\, dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(y|x)\, f_X(x)\, dy\, dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \frac{f(x,y)}{f_X(x)} f_X(x)\; dy\; dx \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y\, f(x,y)\; dy\; dx \\
&= \int_{-\infty}^{\infty} y \left(\int_{-\infty}^{\infty} f(x,y)\; dx\right) dy \\
&= \int_{-\infty}^{\infty} y\, f_Y(y)\; dy = E(Y)
\end{aligned}$$

Similarly,

$$E\big[E(Y|X)\big] = E(X)$$

Example - Family Cars, cont. (Example 4.7.2 deGroot)

We can confirm the Law of Total Probability for Expectations using the data from the previous example.

Table 4.2 Joint p.f. $f(x, y)$ of X and Y in Example 4.7.2 together with marginal p.f.'s $f_1(x)$ and $f_2(y)$

y	x: 1	2	3	4	5	6	7	8	$f_2(y)$
0	0.040	0.028	0.012	0.008	0.008	0.004	0	0	0.100
1	0.048	0.084	0.100	0.120	0.100	0.060	0.020	0.004	0.536
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$f_1(x)$	0.092	0.140	0.164	0.208	0.208	0.120	0.048	0.020	

x	1	2	3	4	5	6	7	8
$E(Y\|X=x)$	0.609	1.057	1.317	1.442	1.538	1.533	1.75	2

$$E(Y) = \sum_{y=0}^{3} y\, f_Y(y) = 0 \times 0.100 + 1 \times 0.536 + 2 \times 0.280 + 3 \times 0.084 = 1.348$$

$$E(Y) = E[E(Y|X)] = \sum_{x=1}^{8} E(Y|X=x)\, f_X(x) = 0.609 \times 0.092 + \cdots + 2 \times 0.020 = 1.348$$

Example - Conditional Uniforms

Let $X \sim \text{Unif}(0,1)$ and $Y|X \sim \text{Unif}(x,1)$, find $E(Y)$.

Example - Deriving Useful Identities

Using conditional expectations in this way is useful as it allows us to treat the random variable we are conditioning on as constant, which can make some problems much simpler.

Let $E(Y|X) = aX + b$ for some constants a and b, what is $E(XY)$?

Conditional Variance

Similar to conditional expectation we can also define conditional variance

$$
\begin{aligned}
Var(Y|X) &= E\left[(Y - E(Y|X))^2 \middle| X\right] \\
&= E(Y^2|X) - E(Y|X)^2
\end{aligned}
$$

there is also an equivalent to the law of total probability for expectations, the Law of Total Probability for Variance:

$$Var(Y) = E[Var(Y|X)] + Var[E(Y|X)]$$

Example - Joint Distribution Exercise deGroot 4.7.7

Suppose that X and Y have a continuous joint distribution for which the joint pdf is as follows:

$$f(x,y) = \begin{cases} x + y & \text{for } 0 < x < 1 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find $E(Y|X)$ and $Var(Y|X)$.