Lecture 12: Covariance / Correlation & Bivariate Normal

Sta 111

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Covariance

We have previously discussed Covariance in relation to the variance of the sum of two random variables (Review Lecture 8).

$$Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y)$$

Specifically, covariance is defined as

$$\begin{aligned} Cov(X,Y) &= E[(X-E(X))(Y-E(Y))] \\ &= E(XY) - E(X)E(Y) \end{aligned}$$

this is a generalization of variance to two random variables and generally measures the degree to which X and Y tend to be large (or small) at the same time or the degree to which one tends to be large while the other is small (positive or negative linear association).

Covariance, cont.

The magnitude of the covariance is not usually informative since it is affected by the magnitude of both X and X. However, the sign of the covariance tells us something useful about the relationship between X and Y.

Consider the following conditions:

- $X > \mu_X$ and $Y > \mu_Y$ then $(X-\mu_X)(Y-\mu_Y)$ will be positive.
- $X < \mu_X$ and $Y < \mu_Y$ then $(X-\mu_X)(Y-\mu_Y)$ will be positive.
- $X > \mu_X$ and $Y < \mu_Y$ then $(X-\mu_X)(Y-\mu_Y)$ will be negative.
- $X < \mu_X$ and $Y > \mu_Y$ then $(X-\mu_X)(Y-\mu_Y)$ will be negative.

Properties of Covariance

- $Cov(X,Y) = E[(X-\mu_x)(Y-\mu_y)] = E(XY) - \mu_x\mu_y$
- $Cov(X,Y) = Cov(Y,X)$
- $Cov(X,Y) = 0$ if X and Y are independent
- $Cov(X,c) = 0$
- $Cov(X,X) = Var(X)$
- $Cov(aX,bY) = ab\ Cov(X,Y)$
- $Cov(X+a,Y+b) = Cov(X,Y)$
- $Cov(X,Y+Z) = Cov(X,Y) + Cov(X,Z)$

Correlation

Since $Cov(X, Y)$ depends on the magnitude of X and Y we would prefer to have a measure of association that is not affected by changes in the scales of the random variables.

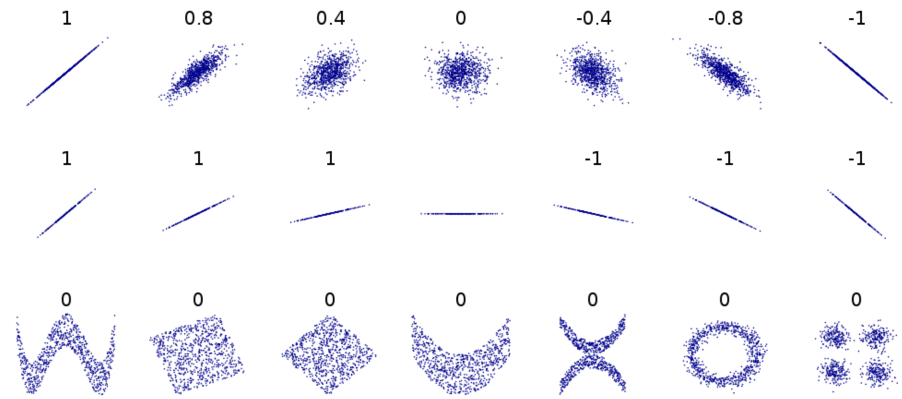
The most common measure of *linear* association is correlation which is defined as

$$\rho(X, Y) = \frac{Cov(X, Y)}{\sigma_X \sigma_Y}$$

$$-1 < \rho(X, Y) < 1$$

Where the magnitude of the correlation measures the strength of the *linear* association and the sign determines if it is a positive or negative relationship.

Correlation, cont.



Correlation and Independence

Given random variables X and Y

$$X \text{ and } Y \text{ are independent} \implies Cov(X, Y) = \rho(X, Y) = 0$$

$$Cov(X, Y) = \rho(X, Y) = 0 \;\;\not\Longrightarrow\;\; X \text{ and } Y \text{ are independent}$$

$Cov(X, Y) = 0$ is necessary but not sufficient for independence!

Example

Let $X = \{-1, 0, 1\}$ with equal probability and $Y = X^2$. Clearly X and Y are not independent random variables.

Example - Linear Dependence

Let $X \sim \text{Unif}(0,1)$ and $Y = aX + b$ for constants a and b. Find $Cov(X,Y)$ and $\rho(X,Y)$

Sums of Normal RVs

If we let $X \sim N(\mu_x, \sigma_x^2)$ and $Y \sim N(\mu_x, \sigma_y^2)$ what is the distribution of $X + Y$?

Hint the MGF for a Normal RV is $\exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$.

General Bivariate Normal

Let $Z_1, Z_2 \sim \mathcal{N}(0,1)$, which we will use to build a general bivariate normal distribution.

$$f(z_1, z_2) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}(z_1^2 + z_2^2)\right]$$

We want to transform these unit normal distributions to have the follow parameters: $\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho$

$$X = \mu_X + \sigma_X Z_1$$
$$Y = \mu_Y + \sigma_Y \left(\rho Z_1 + \sqrt{1-\rho^2}\, Z_2\right)$$

General Bivariate Normal - Marginals

First, lets examine the marginal distributions of X and Y,

General Bivariate Normal - Cov/Corr

Second, we can find $Cov(X,Y)$ and $\rho(X,Y)$

General Bivariate Normal - RNG

Consequently, if we want to generate a Bivariate Normal random variable with $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$ where $\text{Corr}(X,Y) = \rho$ we can generate two independent unit normals Z_1 and Z_2 and use the transformation:

$$X = \mu_X + \sigma_X Z_1$$
$$Y = \mu_Y + \sigma_Y \left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right)$$

General Bivariate Normal - Density

The joint density of X and Y is given by

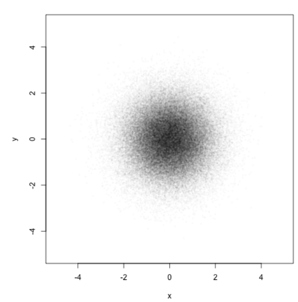
$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y(1-\rho^2)^{1/2}} \exp\left[\frac{-1}{2(1-\rho^2)}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - 2\rho\frac{(x-\mu_X)}{\sigma_X}\frac{(y-\mu_Y)}{\sigma_Y}\right)\right].$$

Alternatively, we can use matrix notation to get a slightly more compact representation

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \boldsymbol{\mu} = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix} \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix}$$

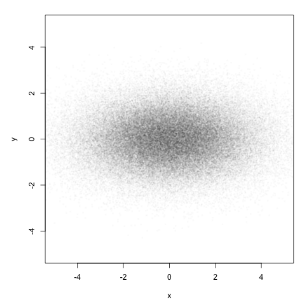
$$f(\boldsymbol{x}) = \frac{1}{2\pi}(\det\boldsymbol{\Sigma})^{-1/2}\exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$$

General Bivariate Normal - Examples



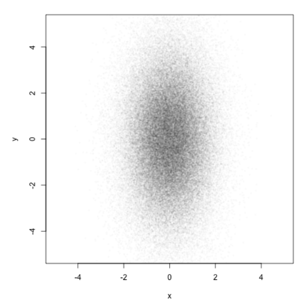
$X \sim \mathcal{N}(0,1),\ Y \sim \mathcal{N}(0,1)$

$\rho = 0$



$X \sim \mathcal{N}(0,2),\ Y \sim \mathcal{N}(0,1)$

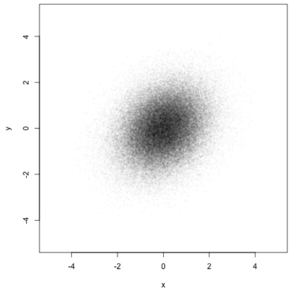
$\rho = 0$



$X \sim \mathcal{N}(0,1),\ Y \sim \mathcal{N}(0,2)$

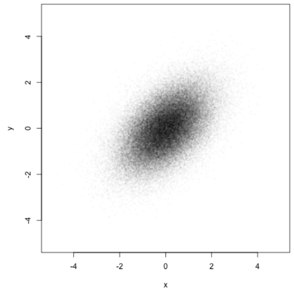
$\rho = 0$

General Bivariate Normal - Examples



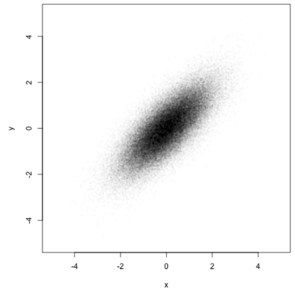
$X \sim \mathcal{N}(0,1),\ Y \sim \mathcal{N}(0,1)$

$\rho = 0.25$



$X \sim \mathcal{N}(0,1),\ Y \sim \mathcal{N}(0,1)$

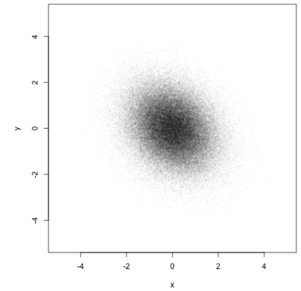
$\rho = 0.5$



$X \sim \mathcal{N}(0,1),\ Y \sim \mathcal{N}(0,1)$

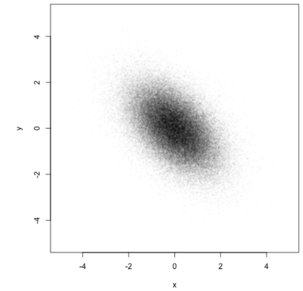
$\rho = 0.75$

General Bivariate Normal - Examples



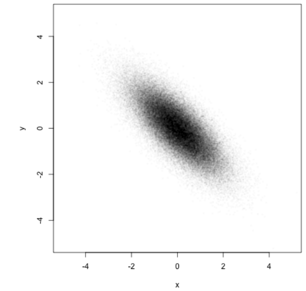
$X \sim \mathcal{N}(0,1),\ Y \sim \mathcal{N}(0,1)$

$\rho = -0.25$



$X \sim \mathcal{N}(0,1),\ Y \sim \mathcal{N}(0,1)$

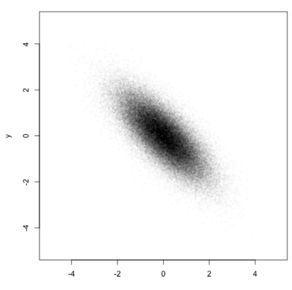
$\rho = -0.5$



$X \sim \mathcal{N}(0,1),\ Y \sim \mathcal{N}(0,1)$

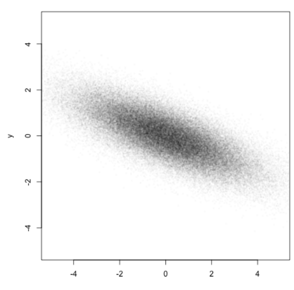
$\rho = -0.75$

General Bivariate Normal - Examples



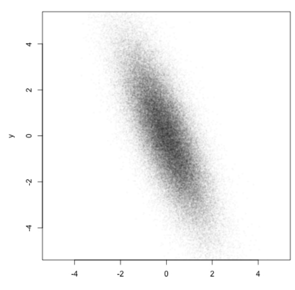
$X \sim \mathcal{N}(0,1),\ Y \sim \mathcal{N}(0,1)$

$\rho = -0.75$



$X \sim \mathcal{N}(0,2),\ Y \sim \mathcal{N}(0,1)$

$\rho = -0.75$



$X \sim \mathcal{N}(0,1),\ Y \sim \mathcal{N}(0,2)$

$\rho = -0.75$

Conditional Expectation of the Bivariate Normal

Using $X = \mu_X + \sigma_X Z_1$ and $Y = \mu_Y + \sigma_Y[\rho Z_1 + (1-\rho^2)^{1/2} Z_2]$ where $Z_1, Z_2 \sim \mathcal{N}(0,1)$ we can find $E(Y|X)$.

Conditional Variance of the Bivariate Normal

Using $X = \mu_X + \sigma_X Z_1$ and $Y = \mu_Y + \sigma_Y[\rho Z_1 + (1-\rho^2)^{1/2} Z_2]$ where $Z_1, Z_2 \sim \mathcal{N}(0,1)$ we can find $Var(Y|X)$.

Example - Husbands and Wives (Example 5.10.6, deGroot)

Suppose that the heights of married couples can be explained by a bivariate normal distribution. If the wives have a mean heigh of 66.8 inches and a standard deviation of 2 inches while the heights of the husbands have a mean of 70 inches and a standard deviation of 2 inches. The correlation between the heights is 0.68. What is the probability that for a randomly selected couple the wife is taller than her husband?