Lecture 9: Gamma Distribution

Sta 111

Colin Rundel

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Exponential Distribution

Let X be a random variable that reflects the time between events which occur continuously with a given rate λ, $X \sim \text{Exp}(\lambda)$

$$f(x|\lambda) = \lambda e^{-\lambda x}$$
$$P(X \le x) = F(x|\lambda) = 1 - e^{-\lambda x}$$

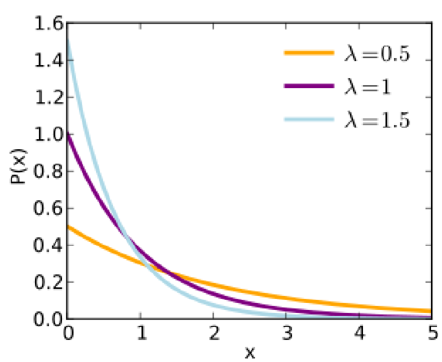
$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} = \left(\frac{\lambda}{\lambda - t}\right)$$

$$E(X) = \lambda^{-1}$$
$$Var(X) = \lambda^{-2}$$
$$Median(X) = \frac{\log 2}{\lambda}$$

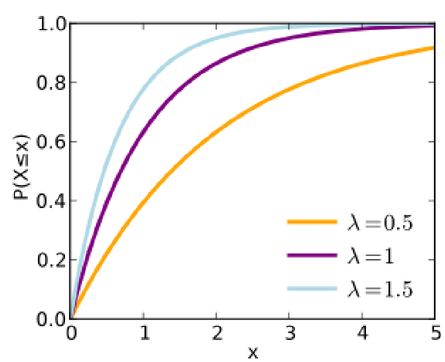
Memoryless property - $P(X > s + t | X > s) = P(X > t)$

Exponential Distribution, cont.

Probability density function:



Cumulative distribution function:



Minimum of Independent Exponential RVs

So far we have just discussed simple functions of random variables involving things like addition, subtraction, multiplication, etc. In many practical applications we may want to know something like the following:

There are two busses whose arrival times have independent exponential distribution with rates λ_1 and λ_2, what is the distribution of the time I have to wait until one of the two busses arrives?

In this example we have the arrival time of bus one given by $X_1 \sim \text{Exp}(\lambda_1)$ and the arrival time of bus two given by $X_2 \sim \text{Exp}(\lambda_2)$ and my waiting time is given by Y where $Y = min(X_1, X_2)$. What is the distribution of X?

Minimum of Independent Exponential RVs, cont.

With a little bit of thought you should be able to see that $Y > a$ can only occur if both $X_1 > a$ and $X_2 > a$ therefore

Raw Moments of Exponential Distribution

We know we can find $E(X^n)$ using the moment generating function but for some distributions we can find a simpler result.

Assume that $n \geq 1$ and $X \sim \text{Exp}(\lambda)$, what is $E[X^n]$?

Generalizing the Factorial

We have just shown the following that when $X \sim \text{Exp}(\lambda)$:

$$E(X^n) = \frac{n!}{\lambda^n}$$

Lets set $\lambda = 1$ and define an new value $\alpha = n + 1$

$$E(X^{\alpha-1}) = (\alpha - 1)!$$
$$\int_0^\infty x^{\alpha-1} e^{-x} dx = (\alpha - 1)!$$
$$\Gamma(\alpha) \equiv \int_0^\infty x^{\alpha-1} e^{-x} dx = (\alpha - 1)!$$

Using a tradition definition of the factorial it only makes sense when $n \in \mathbb{N}$ but we can use this new definition of the gamma function $\Gamma(\alpha)$ for any $\alpha > -1$

Negative Binomial Distribution

Let X be a random variable reflecting the total number of successes before the r^{th} failure where each trial is an independent Bernoulli trial with p probability of success. Then the probability of k successes is given by the Negative Binomial distribution, $X \sim \text{NB}(r, p)$

$$f(k|r,p) = \binom{k+r-1}{k} p^k (1-p)^r$$

Generalizing the Negative Binomial Distribution

We have previously shown a generalization of the factorial function to the postive real numbers ($\mathbb{R}^+$), if we replace the factorial function in the Negative Binomial with the gamma function we can extend the distribution to positive real values of r.

$$\begin{aligned} f(k|r,p) &= \binom{k+r-1}{k} p^k (1-p)^r \\ &= \frac{(k+r-1)!}{k!(r-1)!} p^k (1-p)^r \\ &= \frac{\Gamma(k+r)}{k!\Gamma(r)} p^k (1-p)^r \end{aligned}$$

In which case the distribution is still discrete, but both parameters are now continuous. This variant of the Negative Binomial does not have a natural interpretation as it does not make sense to have 2.537 failures.

Gamma - CDF

Imagine instead of finding the time until an event occurs we instead want to find the distribution for the time until the nth event.

Let T_n denote the time at which the nth event occurs, then $T_n = X_1 + \cdots + X_n$ where $X_1, \ldots, X_n \overset{iid}{\sim} \text{Exp}(\lambda)$.

Gamma/Erlang Distribution - pdf

Erlang Distribution

Let X reflect the time until the nth event occurs when the events occur according to a Poisson process with rate λ, $X \sim \text{Er}(n, \lambda)$

$$f(x|n,\lambda) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}$$

$$F(x|n,\lambda) = \sum_{j=n}^{\infty} \frac{e^{-\lambda x}(\lambda x)^j}{j!}$$

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^n$$

$$\begin{aligned} E(X) &= n/\lambda \\ Var(X) &= n/\lambda^2 \end{aligned}$$

Gamma Distribution - Rate parameterization

We can generalize the Erlang distribution by using the gamma function instead of the factorial function.

$$f(x|n,\lambda) = \frac{\lambda^n}{\Gamma(n)} x^{n-1} e^{-\lambda x}$$
$$F(x|n,\lambda) = \frac{\int_0^x e^{-t/\theta} t^{n-1}\, dt}{\theta^n \Gamma(n)} = \frac{\gamma(n, x/\theta)}{\Gamma(n)}$$

$$M_X(t) = \left(\frac{1}{1 - t/\lambda}\right)^n$$

$$E(X) = n/\lambda$$
$$Var(X) = n/\lambda^2$$

Gamma Distribution - Scale parameterization

We can also sometimes reparameterize using $\theta = 1/\lambda$,

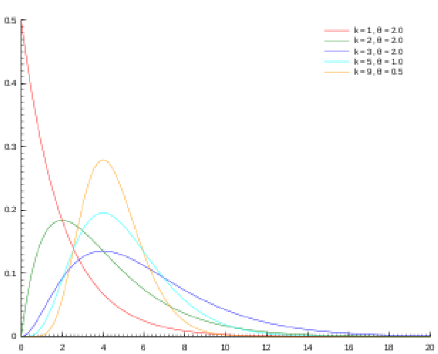
$$f(x|n,\lambda) = \frac{1}{\theta^n \Gamma(n)} x^{n-1} e^{-x/\theta}$$
$$F(x|n,\lambda) = \frac{\int_0^x e^{-t/\theta} t^{n-1}\, dt}{\theta^n \Gamma(n)} = \frac{\gamma(n, x/\theta)}{\Gamma(n)}$$

$$M_X(t) = \left(\frac{1}{1 - \theta t}\right)^n$$

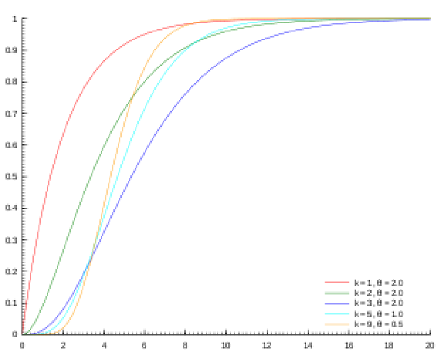
$$E(X) = n\theta$$
$$Var(X) = n\theta^2$$

Gamma Distribution, cont.

Probability density function:



Cumulative distribution function:



Example

Suppose component lifetimes are exponentially distributed with a mean of 10 hours.

Find:

(a) the probability that a component survives 20 hours.

(b) the median component lifetime.

(c) the SD of component lifetime.

(d) the probability the average lifetime of 100 independent components exceeds 11 hours.

(e) the probability the average lifetime of 2 independent components exceeds 11 hours.