Lecture 2: Basic Probability

Sta 230 / Mth 230

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What does it mean to say that:

- The probability of rolling snake eyes is $P(S) = 1/36$?
- The probability of flipping a coin and getting heads is $P(H) = 1/2$?
- The probability Apple's stock price goes up today is $P(+) = 3/4$?

Interpretations:

- Symmetry: If there are k equally-likely outcomes, each has
$$P(E) = 1/k$$
- Frequency: If you can repeat an experiment indefinitely,
$$P(E) = \lim_{n\to\infty} \frac{\#E}{n}$$
- Belief: If you are indifferent between winning \$1 if E occurs or winning \$1 if you draw a blue chip from a box with $100 \times p$ blue chips, rest red,
$$P(E) = p$$

Terminology

Outcome space (Ω) - set of all possible outcomes (ω).

Examples:	3 coin tosses	{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
	One die roll	{1,2,3,4,5,6}
	Sum of two rolls	{2,3,...,11,12}
	Seconds waiting for bus	$[0, \infty)$

Event (E) - subset of Ω $(E \subseteq \Omega)$ that might happen, or might not

Examples:	2 heads	{HHT, HTH, THH}
	Roll an even number	{2,4,6}
	Wait $<$ 2 minutes	$[0, 120)$

Random Variable (X) - a value that depends somehow on chance

Examples:	# of heads	{3, 2, 2, 1, 2, 1, 1, 0}
	# flips until heads	{3, 2, 1, 1, 0, 0, 0, 0}
	2^die	{2, 4, 8, 16, 32, 64}

Event space is *big*

Even for relatively small outcome spaces there are a lot of possible events we can define.

Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ then

$$E = \{\emptyset,\ \omega_1,\ \omega_2,\ \omega_3,\ \omega_1 \cup \omega_2,\ \omega_1 \cup \omega_3, \omega_2 \cup \omega_3,\ \omega_1 \cup \omega_2 \cup \omega_3, \}$$
$$|E| = 8 = 2^3$$

We can show that for an outcome space $\Omega = \{\omega_1, \omega_2, \ldots, \omega_n\}$ the total number of possible events is 2^n.

$$E_1 = \emptyset,\ E_2 = \omega_1,\ E_3 = \omega_2,\ \ldots\ ,\ E_{n+1} = \omega_n,\ E_{n+2} = \omega_1 \cup \omega_2,\ \ldots\ ,\ E_{2^n} = \bigcup_{i=1}^{n} \omega_i$$

Why can we get away with exclusively using unions here?

Rules of Probability - Kolmogorov's axioms

(1) Non-negative:

$$P(E) \geq 0$$

(2) Addition:

$$P(E \cup F) = P(E) + P(F) \text{ if } E \cap F = \emptyset$$

(2)' Countable Addition:

$$P\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} P(E_i) \text{ if } E_iE_j = \emptyset \text{ for } i \neq j$$

(3) Total one:

$$P(\Omega) = 1$$

Useful Identities

Complement Rule:

$$P(\text{not } A) = P(A^c) = 1 - P(A)$$

Difference Rule:

$$P(B \text{ and not } A) = P(BA^c) = P(B) - P(A) \text{ if } A \subseteq B$$

Inclusion-Exclusion:

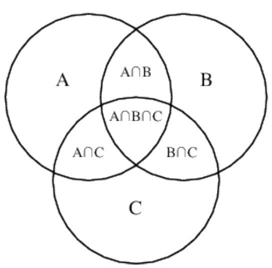
$$P(A \cup B) = P(A) + P(B) - P(AB)$$

Generalized Inclusion-Exclusion

$$P(\bigcup_{i=1}^{n} E_i) = \sum_{i \leq n} P(E_i) - \sum_{i<j\leq n} P(E_iE_j) + \sum_{i<j<k\leq n} P(E_iE_jE_k) - \ldots + (-1)^{n+1}P(E_1 \ldots E_n)$$

For the case of $n = 3$:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A\ B) - P(A\ C) - P(B\ C) + P(A\ B\ C)$$



Equally Likely Outcomes

$$P(E) = \frac{\#(E)}{\#(\Omega)} = \sum_i \frac{1_{\omega_i \in E}}{\#(\Omega)}$$

Notation:

Cardinality - $\#(S)$ = number of elements in set S

Indicator function - $1_{x\in S} = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$

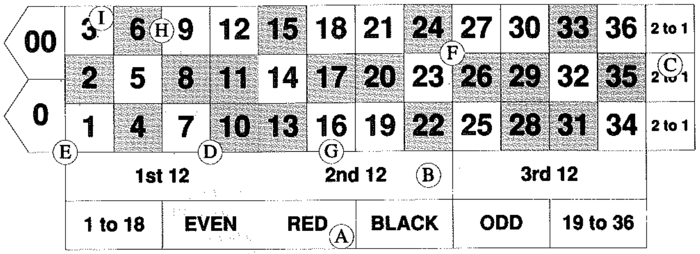
Probability of rolling an even number with a six sided die?

$$E = \{2,4,6\} \text{ and } \Omega = \{1,2,3,4,5,6\}$$

$$P(E) = 3/6 = 1/2$$

Roulette

FIGURE 1. Layout of a Nevada roulette table. Key to colors: 0 and 00 = Green, *unshaded numbers* = Red, *shaded numbers* = Black.



Roulette, cont.

Play	Set of winning numbers	Payoff odds
A. Even money play	Group of 18 numbers as marked in the box	1 to 1
B. Dozen play	12 numbers marked in the box	2 to 1
C. Column play	12 numbers in column (shown here as a row)	2 to 1
D. Line play	Six numbers above	5 to 1
E. House special	0, 00, 1, 2, 3	6 to 1
F. Quarter play	Four numbers in square	8 to 1
G. Street play	Three numbers above	11 to 1
H. Split play	Two adjoining numbers	17 to 1
I. Straight play	Single number	35 to 1

Sampling

Imagine an urn filled with white and black marbles
. . . or a deck of cards
. . . or a bingo cage
. . . or a hat full of raffle tickets

Two common options + one extra for completeness:

- Sampling without replacement
- Sampling with replacement
- Pólya urn model

Birthday Problem

Ignoring leap years, and assuming birthdays are equally likely to be any day of the year, what is the chance of a tie in birthdays among the students in this class?

Some computationally relevant notes

Factorials get really big really quickly, $100! \approx 9.332622 \times 10^{157}$.

Software tends to be quite literal when you ask for something like:

```
factorial(100)/factorial(99)

## [1]100

factorial(200)/factorial(199)

## Warning:  value out of range in 'gammafn'
## Warning:  value out of range in 'gammafn'

## [1] NaN
```


log makes everything better

Remember:

$$\log(XY) = \log(X) + \log(Y)$$
$$\log(X/Y) = \log(X) - \log(Y)$$
$$\log(X^b) = b\log(X)$$

Most often math libraries implement `gamma` and `lgamma` where

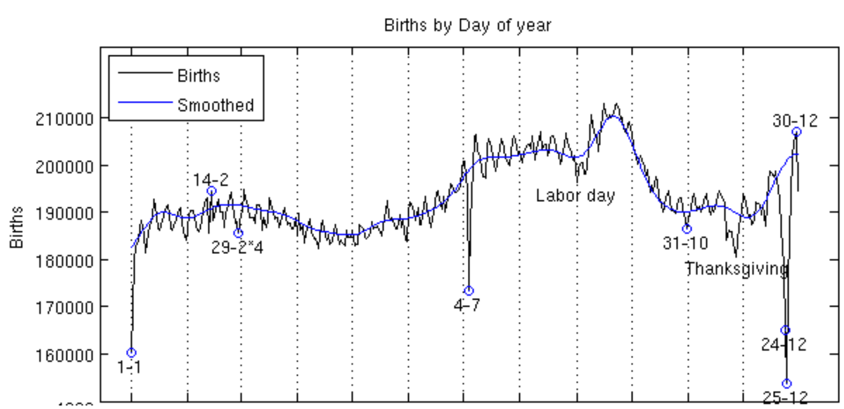
$$\Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt$$

with $\Gamma(n+1) = n!$ when n is a positive integer.

We can rewrite our formula for the birthday problem as

$$\begin{aligned} P(\text{no match w/ } n = n) &= \frac{365!}{(365-n)!\,365^n} \\ &= \exp\left(\log 365! - \log(365-n)! - n\log(365)\right) \\ &= \exp\left(\log\Gamma(366) - \log\Gamma(366-n) - n\log(365)\right) \end{aligned}$$

Birthday Problem, cont.



Something a little more complicated . . .

What is the probably that if you deal yourself 5 cards you will have a royal flush? (Ace, king, queen, jack, and 10 of the same suit)

Conditional Probability

In a setting where we have equally likely outcomes we can define conditional probability as:

For a finite set Ω of equally likely outcomes with events A and B then the *conditional probability* of A *given* B is

$$P(A|B) = \frac{\#(A \cap B)}{\#(B)}$$

or the proportion of outcomes in B that are also in A.

Conditional Probability, cont.

We can rewrite the counting definition of conditional probability as

$$\begin{aligned} P(A|B) &= \frac{\#(A \cap B)}{\#(B)} \\ &= \frac{\#(A \cap B)/\#(\Omega)}{\#(B)/\#(\Omega)} \\ &= \frac{P(A \cap B)}{P(B)} \end{aligned}$$

which is the general definition of conditional probability.

Note that $P(A|B)$ is undefined if $P(B) = 0$.

Useful Rules

Very often we may know the probability of events and their conditional probabilities but not probabilities of the events together, in which case we can use

Multiplication rule:

$$P(A \cap B) = P(A|B)P(B)$$

Other cases where we do not have the probability of one of the events, we can use

Law of total probably:

For a partition $B_1, \ldots, B_n$ of Ω,

$$P(A) = P(A|B_1)P(B_1) + \ldots + P(A|B_n)P(B_n)$$

Independence

We defined events A and B to be independent when

$$P(A \cap B) = P(A)P(B)$$

which also implies that

$$\begin{aligned} P(A|B) &= P(A) \\ P(B|A) &= P(B) \end{aligned}$$

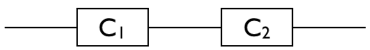
Not to be confused with mutually exclusive events where

$$P(A \cap B) = 0$$

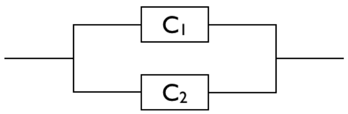
Example, Reliability

If we assume that the probability that a component C_i does not fail in the next week is p_i. What is the probability that the system continues to work for the next week?

Series:



Parallel:



Bayes' Rule

Expands on the definition of conditional probability to give a relationship between $P(B|A)$ and $P(A|B)$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)}$$

In the case where $P(A)$ is not known we can extend this using the law of total probability

$$P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}$$

Example - *House*

If you've ever watched the TV show *House* on Fox, you know that Dr. House regularly states, "It's never lupus."

Lupus is a medical phenomenon where antibodies that are supposed to attack foreign cells to prevent infections instead see plasma proteins as foreign bodies, leading to a high risk of blood clotting. It is believed that 2% of the population suffer from this disease.

The test for lupus is very accurate if the person actually has lupus, however is very inaccurate if the person does not. More specifically, the test is 98% accurate if a person actually has the disease. The test is 74% accurate if a person does not have the disease.

Is Dr. House correct even if someone tests positive for Lupus?

Generalizing Conditional Probability

These rules can be naturally extended to more than one event as follows.

For three events:

$$P(A \cap B \cap C) = P(A \cap B)P(C|A,B) = P(A)P(B|A)P(C|A,B)$$

For n events:

$$P(\cap A_i) = P(A_1)P(A_2|A_1)P(A_3|A_1,A_2)\cdots P(A_n|A_1,\ldots,A_{n-1})$$

Return to the Birthday Problem

What we saw previously saw for the birthday problem,

$$\begin{aligned} P(\text{no match in } n) &= \left(\frac{365}{365}\right)\left(\frac{364}{365}\right)\cdots\left(\frac{365-(n-1)}{365}\right) \\ &= \frac{365!}{(365-n)!\ 365^n} \end{aligned}$$

Let A_i be the event that student i does not match any of the preceding students then

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)P(A_2|A_1)\cdots P(A_n|A_1, A_2, \ldots, A_{n-1})$$

Example - Competing Slots

Imagine two slot machines that have a probability of winning of 50% and 66.6% respectively, but you do not know which machine is which. If we play one machine twice what is the probability that we are playing the good machine given the possible outcomes?

Let W_n be the event of winning on the nth pull and G be the event we played the good machine and B the bad then: