

## Lecture 11: Continuous Random Variables

Sta230 / Mth230

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## Uniform Distribution, from first principals

Lets define a continuous probability distribution that has a constant value  $c$  between  $a$  and  $b$  where  $a \leq b$ . What is the pdf, CDF,  $E(X)$ ,  $\text{Var}(X)$ , mgf?

## Uniform Distribution - CDF

## Uniform Distribution - $E(X)$ , $\text{Var}(X)$

## Uniform Distribution - mgf, raw moments

## Probability density function weirdness

Let  $X = U^2$  where  $U \sim \text{Unif}(0, 1)$  what are the cdf and pdf of  $X$ ?

## Exponential Distribution

## Exponential Distribution, cont.

Let  $X$  be a random variable that reflects the time between events which occur continuously with a given rate  $\lambda$ ,  $X \sim \text{Exp}(\lambda)$

$$f(x|\lambda) = \lambda e^{-\lambda x}$$
$$P(X \leq x) = F(x|\lambda) = 1 - e^{-\lambda x}$$

$$M_X(t) = \left(1 - \frac{t}{\lambda}\right)^{-1} = \left(\frac{\lambda}{\lambda - t}\right)$$

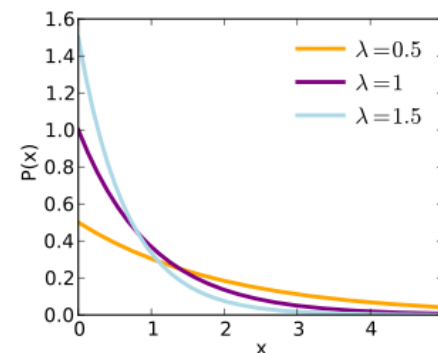
$$E(X) = \lambda^{-1}$$

$$\text{Var}(X) = \lambda^{-2}$$

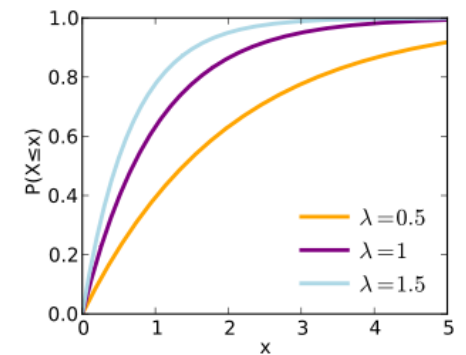
$$\text{Median}(X) = \frac{\log 2}{\lambda}$$

Memoryless property -  $P(X > s + t | X > s) = P(X > t)$

Probability density function:



Cumulative distribution function:



## Minimum of Independent Exponential RVs

So far we have just discussed simple functions of random variables involving things like addition, subtraction, multiplication, etc. In many practical applications we may want to know something like the following:

There are two busses whose arrival times have independent exponential distribution with rates  $\lambda_1$  and  $\lambda_2$ , what is the distribution of the time I have to wait until one of the two busses arrives?

In this example we have the arrival time of bus one given by  $X_1 \sim \text{Exp}(\lambda_1)$  and the arrival time of bus two given by  $X_2 \sim \text{Exp}(\lambda_2)$  and my waiting time is given by  $X$  where  $X = \min(X_1, X_2)$ . What is the distribution of  $X$ ?

## Minimum of Independent Exponential RVs, cont.

With a little bit of thought you should be able to see that  $X > a$  can only occur if both  $X_1 > a$  and  $X_2 > a$  therefore

## Raw Moments of Exponential Distribution

We know we can find  $E(X^n)$  using the moment generating function but for some distributions we can find a simpler result.

Assume that  $n \geq 1$  and  $X \sim \text{Exp}(\lambda)$ , what is  $E[X^n]$ ?

## Generalizing the Factorial

We have just shown the following that when  $X \sim \text{Exp}(\lambda)$ :

$$E(X^n) = \frac{n!}{\lambda^n}$$

Let set  $\lambda = 1$  and define an new value  $\alpha = n + 1$

$$\begin{aligned} E(X^{\alpha-1}) &= (\alpha - 1)! \\ \int_0^{\infty} x^{\alpha-1} e^{-x} dx &= (\alpha - 1)! \\ \Gamma(\alpha) &\equiv \int_0^{\infty} x^{\alpha-1} e^{-x} dx = (\alpha - 1)! \end{aligned}$$

Using a tradition definition of the factorial it only makes sense when  $n \in \mathbb{N}$  but we can use this new definition of the gamma function  $\Gamma(\alpha)$  for any  $\alpha \in \mathbb{R}^+$

## Negative Binomial Distribution

Let  $X$  be a random variable reflecting the total number of successes before the  $r^{\text{th}}$  failure where each trial is an independent Bernoulli trial with  $p$  probability of success. Then the probability of  $k$  successes is given by the Negative Binomial distribution,  $X \sim \text{NB}(r, p)$

$$f(k|r, p) = \binom{k+r-1}{k} p^k (1-p)^r$$

## Generalizing the Negative Binomial Distribution

We have previously shown a generalization of the factorial function to the positive real numbers ( $\mathbb{R}^+$ ), if we replace the factorial function in the Negative Binomial with the gamma function we can extend the distribution to positive real values of  $r$ .

$$\begin{aligned} f(k|r, p) &= \binom{k+r-1}{k} p^k (1-p)^r \\ &= \frac{(k+r-1)!}{k!(r-1)!} p^k (1-p)^r \\ &= \frac{\Gamma(k+r)}{k!\Gamma(r)} p^k (1-p)^r \end{aligned}$$

In which case the distribution is still discrete, but both parameters are now continuous. This variant of the Negative Binomial does not have a natural interpretation as it does not make sense to have 2.537 failures.

## Gamma/Erlang Distribution - CDF

Imagine instead of finding the time until an event occurs we instead want to find the distribution for the time until the  $n^{\text{th}}$  event.

Let  $T_n$  denote the time at which the  $n^{\text{th}}$  event occurs, then  $T_n = X_1 + \dots + X_n$  where  $X_1, \dots, X_n \stackrel{iid}{\sim} \text{Exp}(\lambda)$ . Let  $N(t)$  be the number of events that have occurred at time  $t$ .

$$\begin{aligned} F(t) &= P(T_n \leq t) = P(N(t) \geq n) \\ &= \sum_{j=n}^{\infty} P(N(t) = j) \\ &= \sum_{j=n}^{\infty} \frac{e^{-\lambda t} (\lambda t)^j}{j!} \end{aligned}$$

## Gamma/Erlang Distribution - pdf

## Erlang Distribution

Let  $X$  reflect the time until the  $n$ th event occurs when the events occur according to a Poisson process with rate  $\lambda$ ,  $X \sim \text{Er}(n, \lambda)$

$$f(x|n, \lambda) = \frac{e^{-\lambda x} \lambda^n x^{n-1}}{(n-1)!}$$
$$F(x|n, \lambda) = \sum_{j=n}^{\infty} \frac{e^{-\lambda x} (\lambda x)^j}{j!}$$

$$M_X(t) = \left( \frac{\lambda}{\lambda - t} \right)^n$$

$$E(X) = n/\lambda$$
$$\text{Var}(X) = n/\lambda^2$$

## Gamma Distribution

We can generalize the Erlang distribution by using the gamma function instead of the factorial function, we also reparameterize using  $\theta = 1/\lambda$ ,  $X \sim \text{Gamma}(n, \theta)$ .

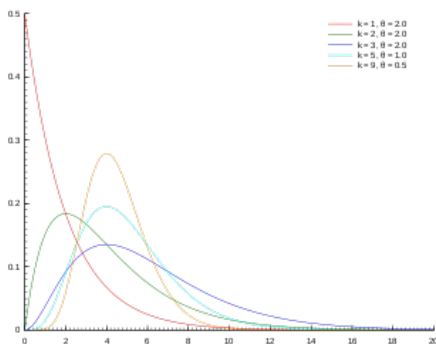
$$f(x|n, \lambda) = \frac{e^{-x/\theta} x^{n-1}}{\theta^n \Gamma(n)}$$
$$F(x|n, \lambda) = \frac{\int_0^x e^{-t/\theta} t^{n-1} dt}{\theta^n \Gamma(n)} = \frac{\gamma(n, x/\theta)}{\Gamma(n)}$$

$$M_X(t) = \left( \frac{1}{1 - \theta t} \right)^n$$

$$E(X) = n\theta$$
$$\text{Var}(X) = n\theta^2$$

## Gamma Distribution, cont.

Probability density function:



Cumulative distribution function:

