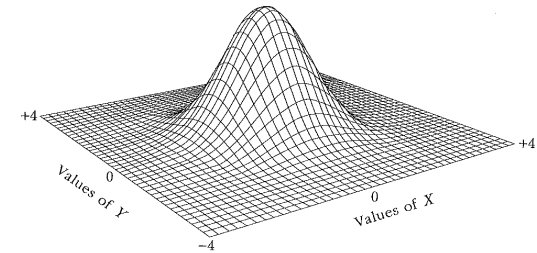


Bivariate Unit Normal

If X and Y have independent unit normal distributions then their joint distribution $f(x, y)$ is given by

$$f(x, y) = \phi(x)\phi(y) = c^2 e^{-\frac{1}{2}(x^2+y^2)}$$



Lecture 17: Bivariate Normal

Sta230 / Mth230

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Bivariate Unit Normal, cont.

We can rewrite the joint distribution in terms of the distance r from the origin

$$r = \sqrt{x^2 + y^2}$$

$$f(x, y) = c^2 e^{-\frac{1}{2}(x^2+y^2)} = c^2 e^{-\frac{1}{2}r^2}$$

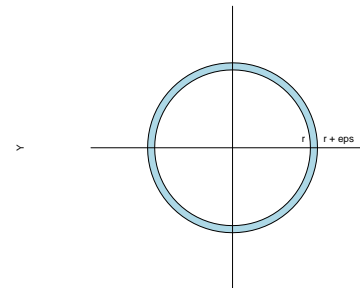
This tells us something useful about this special case of the bivariate normal distributions: it is rotationally symmetric about the origin.

This particular fact is incredibly powerful and helps us solve a variety of problems.

Bivariate Unit Normal - Normalizing Constant

First, rewriting the joint distribution in this way also makes it possible to evaluate the constant of integration, c .

From our previous discussions of the normal distribution we know that $c = \frac{1}{\sqrt{2\pi}}$, but we never saw how to derive it explicitly.



Bivariate Unit Normal - Normalizing Constant

Rayleigh Distribution

The distance from the origin of a point (x, y) derived from $X, Y \sim \mathcal{N}(0, 1)$ is called the Rayleigh distribution.

$$f_R(r) = re^{-\frac{1}{2}r^2}$$

$$\begin{aligned} F_R(r) &= \int_0^r re^{-\frac{1}{2}r^2} dr \\ &= -e^{-\frac{1}{2}r^2} \Big|_0^r \\ &= 1 - e^{-\frac{1}{2}r^2} \end{aligned}$$

We will see the expected value in a little bit.

Rayleigh Distribution - Example

If the position of a target shooter's shots are described by normal distributions in the x and y direction that are centered on the bull's eye with a standard deviation of 1 in:

- What is the probability a shot is more within 1 inch of the bull's eye?
- What is the probability a shot is more than 2 inches away from the bull's eye?
- What is the probability a shot is between 1 and 2 inches from the bull's eye?

Bivariate Unit Normal - Variance of the Normal

Assume we did not know that $\text{Var}(X) = \sigma^2 = 1$, what would we need to evaluate to find the variance of $X \sim \mathcal{N}(0, 1)$?

$$\text{Var}(X) = E(X^2) - E(X)^2 = E(X^2) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2}x^2} dx$$

We can use the Rayleigh distribution to make our life easier

$$\begin{aligned} E(R^2) &= E(X^2 + Y^2) \\ &= E(X^2) + E(Y^2) \\ &= 2E(X^2) \end{aligned}$$

$$E(X^2) = E(R^2)/2$$

Bivariate Unit Normal - Variance of the Normal

$E(R^2)$ is still not very easy to evaluate, but we can consider a change of variables where we let $S = R^2$.

Sums of Independent Normal Variables

Previously we have shown that the sum (or average) of iid random variables will converge to the normal distribution as the number of random variables goes to infinity thanks to the central limit theorem.

Intuitively it makes sense then that the sum of a small number of normals should also be normal but we have not explicitly shown this to be true. Also, what about the case where the normals are not identically distributed?

Based on arguments using projections and the Bivariate Unit Normal's rotational symmetry the book derives the following general result (and special cases):

- i) If $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y \sim \mathcal{N}(\lambda, \tau^2)$, then $X + Y \sim \mathcal{N}(\mu + \lambda, \sigma^2 + \tau^2)$.
- ii) If $X, Y \sim \mathcal{N}(0, 1)$, then $\alpha X + \beta Y \sim \mathcal{N}(0, 1)$ for all α and β with $\alpha^2 + \beta^2 = 1$
- iii) If $X, Y \sim \mathcal{N}(0, 1)$, then $X + Y \sim \mathcal{N}(0, 2)$

Rayleigh Distribution - Expectation

We can also use the preceding result to help find the expected value of R .

Alternative Approach

It is also possible to prove these results using moment generating functions.

Recall:

- Moment generating functions uniquely identify distributions.
- If X and Y are independent with mgfs $M_X(t)$ and $M_Y(t)$ then

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

- If $Z \sim \mathcal{N}(\mu, \sigma^2)$ then

$$M_Z(t) = \exp(\mu t + \frac{1}{2}\sigma^2 t^2)$$

Proof of iii)

Let $X, Y \sim \mathcal{N}(0, 1)$ then

Proof of i)

Let $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y \sim \mathcal{N}(\lambda, \tau^2)$ then

Proof of ii)

Further Generalization and Example

If X, Y, Z are all independent normal random variables then we have shown that $X' = X + Y$ must also be a normal random variable and it follows that $Z' = X' + Z$ must also be a normal random variable, which is equivalent to $X + Y + Z$ which is therefore a normal random variable.

Example - If $X, Y, Z \sim \mathcal{N}(0, 1)$ then what is $P(X + Y < Z + 2)$?

Multivariate Unit Normal

Let $Z_1, \dots, Z_n \sim \mathcal{N}(0, 1)$ then the joint distribution of (z_1, \dots, z_n) is

$$\begin{aligned} f(z_1, \dots, z_n) &= f(z_1) \times \dots \times f(z_n) \\ &= \left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left[-\frac{1}{2}(z_1^2 + \dots + z_n^2)\right] \end{aligned}$$

Once again we can model the distance of a point from the origin in this n -dimensional space using

$$\begin{aligned} R_n &= \sqrt{Z_1^2 + \dots + Z_n^2} \\ P(R_n \in (r, r + \epsilon)) &= c_n \epsilon r^{n-1} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}r^2} \end{aligned}$$

where c_n is the $(n - 1)$ -dimensional volume of the “surface” of a sphere of radius 1 in n -dimensions. Eg. $c_2 = 2\pi$ and $c_3 = 4\pi$

Pearson's χ^2 test

Why do we care about the χ^2 distribution? Imagine we want to decide if a six sided die is fair, we roll the die 100 times and total up the results

Roll	1	2	3	4	5	6
Counts	18	13	17	13	16	23

In 1900 Carl Pearson showed that

$$\sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i} = \sum_{i=1}^k \frac{(N_i - np_i)^2}{np_i} \sim \chi^2(df = k - 1)$$

If you've taken an introductory statistics course you've seen this in the form of a hypothesis test where you calculate a p-value (probability of getting the observed value or more extreme given H_0 is true).

$$\text{p-value} = 1 - F_{\chi_{k-1}^2} \left(\sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i} \right)$$

Distribution of R_n

$$f_{R_n}(r) = c_n r^{n-1} \left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}r^2}$$

If we then perform a change of variables such that $S_n = R_n^2$

$$\begin{aligned} f_{S_n}(s) &= f_{R_n}(r) \left| \frac{dS_n}{dR_n} \right| \\ &= c_n' \frac{r^{n-1} e^{-\frac{1}{2}r^2}}{2r} = \frac{c_n'}{2} r^{n-2} e^{-\frac{1}{2}r^2} \\ &= \frac{c_n'}{2} s^{n/2-1} e^{-s/2} \end{aligned}$$

Which is the Gamma distribution with $k = n/2$ and $\theta = 2$.

$$S_n \sim \text{Gamma}(n/2, 2)$$

This special case of the Gamma distribution is known as the χ^2 distribution with n -degrees of freedom.

Finding c_n

If $X \sim \text{Gamma}(k, \theta)$ then its density is given by

$$f_X(x|k, \theta) = \frac{1}{\theta^k} \frac{1}{\Gamma(k)} x^{k-1} e^{-x/\theta}$$

From this we can find,

Implications of R_n distribution

We have shown that the density for R_n is given by

$$f_{R_n}(r) = c'_n r^{n-1} e^{-\frac{1}{2}r^2}$$

what does this tell us about the distribution of probability mass for a multivariate normal distribution?

An interesting question - where are we most likely to see an observation from an n dimension multivariate normal distribution?